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ROBUST STABILITY AND EVALUATION OF THE QUALITY FUNCTIONAL FOR LINEAR CONTROL SYSTEMS WITH MATRIX UNCERTAINTY

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Consider a continuous linear dynamical control system:

$$\dot{x} = (A + \Delta A(t))x + (B + \Delta B(t))u\,,\;\; y = Cx + Du\,, \qquad (1)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^l$ are state, control, and observable object output vectors respectively, A, B, C and D are constant matrices of corresponding sizes $n \times n$, $n \times m$, $l \times n$ and $l \times m$, and $\Delta A(t) = F_A \Delta_A(t) H_A$, $\Delta B(t) = F_B \Delta_B(t) H_B$ are system uncertainties, where F_A, F_B, H_A, H_B are constant matrices of corresponding size and matrices uncertainties $\Delta_A(t)$ and $\Delta_B(t)$ satisfy the constraints

$$\|\Delta_A(t)\| \le 1\,,\;\; \|\Delta_B(t)\| \le 1 \;\text{ or }\; \|\Delta_A(t)\|_F \le 1\,,\;\; \|\Delta_B(t)\|_F \le 1\,,\;\; t \ge 0\,.$$

Hereinafter, $\|\cdot\|$ is Euclidean vector norm and spectral matrix norm, $\|\cdot\|_F$ is matrix Frobenius norm, I_n is the unit $n \times n$ matrix, $X = X^{\mathrm{T}} > 0$ (≥ 0) is a positive (nonnegative) definite symmetric matrix.

We control the system (1) with output feedback:

$$u = Ky\,,\;\; K = K_0 + \tilde{K}\,,\;\; \tilde{K} \in \mathrm{E}\,, \tag{2}$$

where E is an ellipsoidal set of matrices $\mathrm{E} = \left\{ K \in \mathbb{R}^{m\times l} : K^{\mathrm{T}} P K \le Q \right\}$, where $P = P^{\mathrm{T}} > 0$ and $Q = Q^{\mathrm{T}} > 0$ are symmetric positive definite matrices of corresponding sizes $m \times m$ and $l \times l$.

We introduce on the set of matrices $\mathrm{K}_D = \{ K : \det(I_m - KD) \ne 0 \}$ a nonlinear operator

$$\mathrm{D}\colon \mathbb{R}^{m\times l} \to \mathbb{R}^{m\times l},\;\; \mathrm{D}(K) = (I_m - KD)^{-1} K \equiv K(I_l - DK)^{-1}\,.$$

Theorem 1. *Suppose that for a positive definite matrix* $X = X^{\mathrm{T}} > 0$ *and for some* $\varepsilon_1, \varepsilon_2 > 0$ *the matrix inequalities* $\Delta = D^{\mathrm{T}} Q D - G^{\mathrm{T}} P G < 0$, $G = I_m - K_0 D$ *and*

$$\begin{bmatrix} \Omega & XB + \varepsilon_2 C^{\mathrm{T}} \mathrm{D}^{\mathrm{T}}(K_0) H_B^{\mathrm{T}} H_B & C_0^{\mathrm{T}} & XF_A & XF_B \\ B^{\mathrm{T}} X + \varepsilon_2 H_B^{\mathrm{T}} H_B \mathrm{D}(K_0) C & -G^{\mathrm{T}} P G + \varepsilon_2 H_B^{\mathrm{T}} H_B & D^{\mathrm{T}} & 0 & 0 \\ C_0 & D & -Q^{-1} & 0 & 0 \\ F_A^{\mathrm{T}} X & 0 & 0 & -\varepsilon_1 I & 0 \\ F_B^{\mathrm{T}} X & 0 & 0 & 0 & -\varepsilon_2 I \end{bmatrix} < 0$$

holds, where $C_0 = C + D\mathrm{D}(K_0)C$,

$$\Omega = (A + B\mathrm{D}(K_0)C)^{\mathrm{T}} X + X(A + B\mathrm{D}(K_0)C) + \varepsilon_1 H_A^{\mathrm{T}} H_A + \varepsilon_2 C^{\mathrm{T}} \mathrm{D}^{\mathrm{T}}(K_0) H_B^{\mathrm{T}} H_B \mathrm{D}(K_0) C\,.$$

Then any control (2) ensures asymptotic stability of the zero state for system (1) and the general Lyapunov function $v(x) = x^{\mathrm{T}} X x$.

Consider a control system (1), (2) with quadratic quality functional

$$J(u, x_0) = \int_0^\infty \varphi(x,u)\,dt,\;\; \varphi(x,u) = \begin{bmatrix} x^{\mathrm{T}} & u^{\mathrm{T}} \end{bmatrix} \Phi \begin{bmatrix} x \\ u \end{bmatrix},\;\; \Phi = \begin{bmatrix} S & N \\ N^T & R \end{bmatrix} > 0\,, \tag{3}$$

where x_0 is initial vector, $S = S^{\mathrm{T}} > 0$, $R = R^{\mathrm{T}} > 0$, and N given constant matrices.

We need to describe the set of controls (2) that would provide asymptotic stability for the state $x \equiv 0$ of system (1) and a bound

$$J(u, x_0) \le \omega\,, \tag{4}$$

where $\omega > 0$ is some maximal admissible value of the functional.

Theorem 2. [1, p. 61] *Suppose that for a positive definite matrix* $X = X^{\mathrm{T}} > 0$ *and for some* $\varepsilon_1, \varepsilon_2 > 0$ *the matrix inequalities* $x_0^{\mathrm{T}} X x_0 \le \omega$ *and*

$$G^{\mathrm{T}} PG - D^{\mathrm{T}} QD > R, \tag{5}$$

$$\begin{bmatrix} Z & N_0 & C_0^{\mathrm{T}} & XF_A & XF_B \\ N_0^{\mathrm{T}} & R - G^{\mathrm{T}} PG & D^T & 0 & 0 \\ C_0 & D & Q^{-1} & 0 & 0 \\ F_A^{\mathrm{T}} X & 0 & 0 & -\varepsilon_1 I & 0 \\ F_B^{\mathrm{T}} X & 0 & 0 & 0 & -\varepsilon_2 I \end{bmatrix} < 0 \tag{6}$$

holds, where

$$Z = (A + B\mathrm{D}(K_0)C)^{\mathrm{T}} X + X(A + B\mathrm{D}(K_0)C) + L_0^{\mathrm{T}} \Phi L_0 + \varepsilon_1 H_A^{\mathrm{T}} H_A + \varepsilon_2 C_*^{\mathrm{T}} C_*,$$

$$C_0 = C + D\mathrm{D}(K_0)C, \; L_0^{\mathrm{T}} = \left[I_n \quad C^{\mathrm{T}} \mathrm{D}^{\mathrm{T}}(K_0) \right],$$

$$N_0 = XB + N + C^{\mathrm{T}} \mathrm{D}^{\mathrm{T}}(K_0) R + \varepsilon_2 C_*^{\mathrm{T}} H_B, \; C_* = H_B \mathrm{D}(K_0) C.$$

Then any control (2) ensures asymptotic stability of the zero state for system (1), the general Lyapunov function $v(x) = x^{\mathrm{T}} Xx$*, and a bound on the functional (4).*

Numerical experiment. Consider a control system for a double oscillator. It is system of two solids that connected by a spring and slide without a friction along of horizontal rod (Fig. 1). This system is defined with two linear differential equations of order two, or, in vector-matrix form:

$$\dot{x} = (A + \Delta A(t))x + Bu, \tag{7}$$

where

$$A + \Delta A(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{k_0}{m_1} & \dfrac{k_0}{m_1} & 0 & 0 \\ \dfrac{k_0}{m_2} & -\dfrac{k_0}{m_2} & 0 & 0 \end{bmatrix} + F_A \Delta(t) H_A, \; B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \; F_A = \begin{bmatrix} 0 \\ 0 \\ -\delta \\ \delta \end{bmatrix},$$

$$H_A = [1 \quad -1 \quad 0 \quad 0], \; x = [x_1 \quad x_2 \quad \dot{x}_1 \quad \dot{x}_2]^{\mathrm{T}}.$$

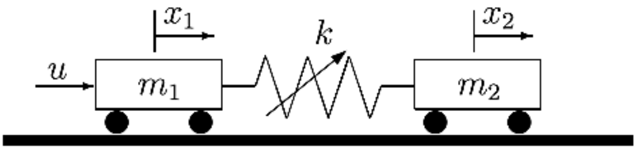

Fig. 1. A two-masse mechanical system

Here x_1 and $\dot{x}_1$ are coordinate and velocity respectively for the first solid, x_2 and $\dot{x}_2$ are coordinate and velocity respectively for the second solid, m_1 and m_2 are masses of the first and second solids respectively. We define a stiffness coefficient as variable periodic function of time $k = k_0 + \delta\Delta(t)$, where $\Delta(t) = \sin(\varpi t)$, $\delta << 1$ is the amplitude of harmonic oscillations, and ϖ is the frequency parameter.

Let $m_1 = 1$, $m_2 = 1$, $k_0 = 2$, $\delta = 0,02$, $\Delta(t) = \sin(t / 5)$.

We assume that the output vector

$$y = Cx + Du = \begin{bmatrix} \dot{x}_1 + u \\ x_2 \end{bmatrix},\ C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},\ D = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

can be measured. We find control in the form static output feedback $u = Ky$, where $K = [k_1 \quad k_2] = K_0 + \tilde{K}$. We find the vector $K_0 = [1,6938 \quad 0,1089]$ that ensures asymptotic stability for system $\dot{x} = (A + B\mathrm{D}(K_0)C)x$,. Here the spectrum equals $\sigma(M_0) = \{-0,3259 \pm 1,6913i;\ -0,8333; -0,3296\}$. The behavior of solutions of system with matrix uncertainty (7) with control $u = K_0 y$ and initial vector $x_0 = [1 \quad -2 \quad 0 \quad 2]^{\mathrm{T}}$.

For demonstration of Theorem 2 we define a matrix functional (3): $S = 0,1I_4$, $R = 0,01$, $N^T = [0,01 \quad 0 \quad 0 \quad 0,01]$. Using the Matlab suite, we find $P = 39,5751$ and positive definite matrices

$$Q = \begin{bmatrix} 29,6673 & 11,8521 \\ 11,8521 & 17,0069 \end{bmatrix},\ X = \begin{bmatrix} 114,3996 & -49,8299 & 42,2629 & 37,6176 \\ -49,8299 & 55,1945 & -15,0903 & -5,6888 \\ 42,2629 & -15,0903 & 36,7322 & 15,6111 \\ 37,6176 & 5,6888 & 15,6111 & 39,8411 \end{bmatrix},$$

that satisfy the inequalities (5), (6) for $\varepsilon_1 = 0,01$.

Thus, for all values of the vector of feedback amplification coefficients $K = K_0 + \tilde{K}$ from a closed region E_0 bounded by the ellipse $(K - K_0)Q^{-1}(K - K_0)^{\mathrm{T}} \leq P^{-1}$, the motion of the system of two solids in a neighborhood of the zero state is asymptotically stable. Here $v(x) = x^{\mathrm{T}} X x$ is a general Lyapunov function, and the value of the given quality functional does not exceed $v(x_0) = 889,8436$.

The results are obtained based on the known generalizations statement on adequacy of Petersen's lemma about matrix uncertainties [2, p. 355], [3, p. 253].

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