# APPLICATION EXAMPLES TO PROBLEMS OF MODERN MATHEMATICAL APPARATUS

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***Abstract.*** *Purpose.* The purpose of the research work is:

– possibility to combine different ways of solving certain mathematical problems. In general, the term "non-standard" methods of solving problems in mathematics has not defined yet, but many authors use this term in their researchers. It should be noted that there are many school problems that use unusual considerations. These are the tasks that are considered to be more complex and require non-standard methods of solving. These methods illustrate the wide possibilities of using well-acquired school knowledge and instill skills in using non-standard methods of reasoning in solving problems;

– the performance of comparative analysis in the calculations of spent work with different adjacent geometry of three-dimensional figures, which are given in this work;

– establishing a mathematical law for calculating the maximum number of embeddings of a set of homogeneous circular objects inside a certain external geometric structure.

*Methodology.* Research of this work is based on the use of modern mathematics, such as school and analytical geometry, the basics of integral calculus and their practical application, progression.

*Practical implications.* The first part of the paper presents several different methods for solving one geometric (stereometric) problem using both elementary geometry and higher mathematics, in particular, analytical geometry. These different ways of solving one specific problem demonstrate the versatility of the modern mathematical apparatus, link the mutual goals and methods of elementary and higher mathematics in specific applications. It is shown that the problem with the school formulation of the condition can

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be solved by means of higher mathematics with the use of actions on vectors, the use of types of products of vectors and so on. The second part considers mathematical models of a specific technical problem based on the known law of physics, such as the calculation of some work under the action of gravity in a particular case. As such work, as it is established, is differentiated on a certain independent variable, therefore at first the value of a separate element of such work is established and integration of this element on this variable within its limits is executed. The given models with different geometry of the location of the three-dimensional body (reservoir, etc.), for which the study of this numerical characteristic is studied, are studied and compared, a comparative analysis is made in these two different positions of this body. The third part investigates the maximum filling of certain geometric both flat and three-dimensional external structures with many circular (spherical) figures, establishes a mathematical law for calculating the quantitative characteristics of such maximum filling, proposed and tested the criterion (coefficient) of efficiency and usefulness of such filling. For each of the parts of the given researches the corresponding figures, tables which supplement accordingly received results in the form of formulas and calculations are offered. Conclusions have been also made on the research conducted in the work.

*Value/originality.* The originality of the research is as follows:

– use and combination of school and higher mathematics in solving a specific geometric problem;

– the use of mathematical apparatus in calculating the work of solving a physical problem;

– establishment of the mathematical law of the maximum filling of a certain geometric structure by a homogeneous set of circular figures.

By solving one geometric problem in different methods or ways, it is possible to better understand the specific method, its advantages and disadvantages depending on the content of the problem. The use of different methods of such a solution provides an opportunity to replace it with another solution, which encourages to find alternative effective creative approaches to solving this problem.

It is not necessary to solve each problem in different ways or methods, just to choose one or two. In order to enhance cognitive activity and learn different methods of solving geometric problems, it is proposed to use non-standard methods of solving geometric problems.

Regarding the amount of work spent, the main element of novelty is the comparative analysis of such calculations in two related cases of the location of a geometric body (cylinder): when its foundations are vertical or horizontal.

The main element of the novelty of the results of the maximum filling of one geometric structure by a set of circular objects is the establishment of the mathematical law of quantitative characteristics of such filling, while proposing and testing a logical convenient coefficient of usefulness of such maximum filling.

## 1. Introduction

We can successfully translate the language spoken by nature into the language of mathematics and understand the structure of the relationships of any phenomenon. And once we formalize these connections, we can build certain mathematical models, predict the future states of the phenomena that these models describe, only on paper or inside computer memory. At the heart of modern mathematics are operations of counting, measuring and describing the shapes of the object under study. Translated from the ancient Greek, its name means "science", "study". This is the basis on which knowledge of structure, order and relations is based. They are the essence of science. Einstein, when asked where his lab was, smiled and pointed to a pencil and paper. Therefore, the role of mathematics is too important in human life, of course, this has not always been the case, people used to do without it, but modern man can not do without calculations of various types of complexity in today's world.

Thus, the main purpose of the research of this work is the possibility and necessity of using a modern mathematical apparatus to solve specific mathematical technical or economic problems and to analyze the theoretical results of these studies. In mathematics, geometric problems play an important and multifaceted role. Solving such problems serves to achieve the goals set by the study of mathematics in both high school and higher education. Therefore, a lot of time in the study of mathematics is devoted to solving geometric problems. Such tasks allow to master the most important mathematical concepts, master mathematical symbolism, teach to perform proofs of various hypotheses, formulas, theorems and statements.

The purpose of the research work is to demonstrate:

– versatility of modern mathematical apparatus on the example of solving one specific mathematical problem;

– use of the known physical law of calculation of work in one applied problem in various variants of placement of a geometric working body;

– study of the maximum filling of certain geometric structures with many other homogeneous bodies (figures).

Mathematical problems of different directions can serve as a foundation for mastering new theoretical issues, consolidate acquired knowledge, illustrate the practical application of the studied material in new approaches in solving certain problems. Thus, the skills and abilities of a certain mental activity are formed, which is combined with such important traits of character as persistence, attentiveness, concentration. The well-known law of physics on the calculation of work under the action of gravity on a certain displacement has some application in another problem of the results of research data, which leads to the formation and calculation of the corresponding integral expressions. The question of the optimal maximum filling of certain external geometric structures with a homogeneous internal set of circular objects is also always an important and urgent task.

*Review of recent research and publications.* To study methods of proving and solving problems of a certain class, it is important to have algorithms, schemes and to understand the outlines of their applications [1]. The application of acquired mathematical knowledge must be found in solving standard and non-standard problems, to predict the results of certain studies. Therefore, at the beginning of solving the problem analyze certain ideas and methods, using illustrative tools, compare all ways of finding a solution to the problem and choose the most logical of them.

In geometry, it is important to form the need for proofs using a variety of methods and ways to solve such problems. When solving a certain problem in several ways or methods, the acquired skills are transferred to other conditions, repeated in new connections [2–3].

In many mathematics applications and the use of mathematical apparatus in solving physical or technical problems, the question of evaluation, calculation of work spent is an extremely important issue [3–6]. The solution of one of such problems in the given two adjacent cases is offered in this work.

The solution of the problem of maximum filling of certain external geometric structures with a homogeneous set of circular (spherical) objects is widely used in various spheres of human life [7–12]. Finding a quantitative characteristic of such a filling is given in these studies.

## 2. Several approaches to solving one geometric problem

Here is an example of one of these problems. It demonstrates methods to solve them.

**Formulation of the problem** is to determine the angle between the bisectors of two plane angles of a regular tetrahedron, which are drawn from one vertex.

**Methods to solve the problem**.

*The first method.*

By the definition of a regular tetrahedron [3], whichever face we consider, we have an equilateral triangle. Drop from the vertex D of the tetrahedron DABC bisectors DM and DN on the face ADC and ADB (Figure 1). We obtain the angle between the bisectors of two plane angles.

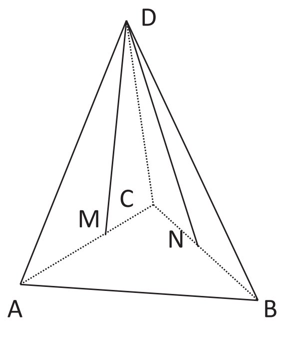


**Figure 1. DABC tetrahedron with the desired angle between the apophemes DM and DN**

Suppose $BC = x$, then $MN = \frac{1}{2}x$ (the middle line of the triangle ABC), $DN = DM = \frac{\sqrt{3}}{2}x$ (by the property of bisectors of equilateral triangles ADC, ADB) and $\angle NDM = \alpha$.

$\varnothing NDM$ – isosceles, where – the height of the triangle and $\angle NDK = \frac{\alpha}{2}$ and $KN = \frac{1}{4}x$, by the definition $\sin\frac{\alpha}{2} = \frac{KN}{DN} = \frac{1}{2\sqrt{3}}$; $\cos\frac{\alpha}{2} = \frac{DK}{DN} = \frac{\sqrt{DN^2 - KN^2}}{DN} = \sqrt{\frac{11}{12}}$.

Then, $\cos\alpha = \cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2} = \frac{11}{12} - \frac{1}{12} = \frac{5}{6}$, where $\alpha = \arccos\frac{5}{6}$.

It should be noted that $\cos\alpha$ can be found by the theorems of cosines and sines:

By the sine theorem: $\frac{DN}{\sin\angle DMN} = \frac{MN}{\sin\angle MDN}$, where $DN = \frac{\sqrt{3}}{2}x$, $\angle DMN = \angle DMK = 90^\circ - \frac{\alpha}{2}$, $MN = \frac{1}{2}x$.

$$\frac{\frac{\sqrt{3}}{2}x}{\sin(90 - \frac{\alpha}{2})} = \frac{\frac{1}{2}x}{\sin\alpha},$$

$$\frac{\sqrt{3}}{\cos\frac{\alpha}{2}} = \frac{1}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}},$$

$$\sin\frac{\alpha}{2} = \frac{1}{2\sqrt{3}},$$

$$\cos\frac{\alpha}{2} = \frac{DK}{DN} = \frac{\sqrt{DN^2 - KN^2}}{DN} = \frac{\sqrt{\frac{3}{4}x^2 - \frac{1}{16}x^2}}{\frac{\sqrt{3}}{2}x} = \sqrt{\frac{11}{12}},$$

$\cos\alpha = \cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2} = \frac{11}{12} - \frac{1}{12} = \frac{5}{6}$, where $\alpha = \arccos\frac{5}{6}$.

By the cosine theorem:

$$MN^2 = DM^2 + DN^2 - 2DM \cdot DN\cos\angle MDN = 2DN^2 - 2DN^2\cos\angle MDN$$

$$\frac{1}{4}x^2 = 2\frac{3}{4}x^2 - 2\frac{3}{4}x^2\cos\alpha,$$

$\cos\alpha = \frac{5}{6}$, де $\alpha = \arccos\frac{5}{6}$.

*The second method.* Place the correct tetrahedron DABC in the Cartesian coordinate system, so that the vertex D lies on the positive half-axis and is projected in the center of the base Δ ABC, located in the XOY plane, ie at the p. (0; 0; 0) (Figure 2).

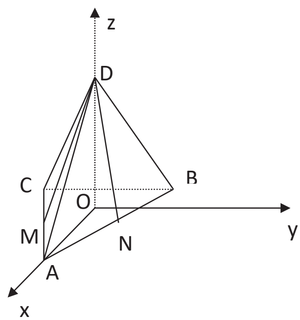


**Figure 2. The given tetrahedron in three-dimensional space, the base center of which coincides with the origin.**

Suppose p. A (1; 0; 0), then p. B (x1; y1; 0) and p. C (x2; y2; 0).

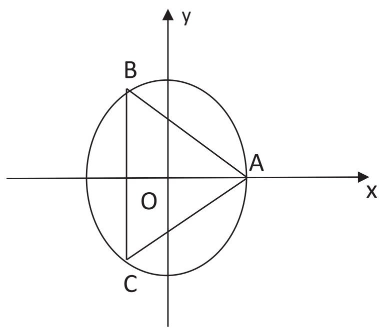


**Figure 3. The base of the tetrahedron in the XOY plane, inscribed in a circle with the center at the origin**

Describe around $\Delta$ABC a unit circle and find the coordinates of points B and C with the definition of trigonometric functions (Figure 3).

$$x_1 = -\cos 60^\circ = -\frac{1}{2};$$
$$y_1 = \sin 60^\circ = \frac{\sqrt{3}}{2}; \quad \text{ie p. } B(-\frac{1}{2};\frac{\sqrt{3}}{2};0).$$
$$x_2 = -\cos 60^\circ = -\frac{1}{2};$$
$$y_2 = -\sin 60^\circ = -\frac{\sqrt{3}}{2}; \quad \text{ie p. } C(-\frac{1}{2};-\frac{\sqrt{3}}{2};0).$$

According to the formula for the distance between two points:

$$AB = \sqrt{\left(-\frac{1}{2}-1\right)^2 + \left(\frac{\sqrt{3}}{2}-0\right)^2 + (0-0)^2} = \sqrt{3}.$$

By placement in space p. $D(0;0;z)$, тоді $\overrightarrow{AD} = \{-1;0;z\}$, $\left|\overrightarrow{AD}\right| = \sqrt{1+z^2}$ .

Since $AD = AB = \sqrt{3}$, then we get equality $\sqrt{1+z^2} = \sqrt{3}$; out of here $z_1 = \sqrt{2}$ and $z_2 = -\sqrt{2}$ (does not satisfy the condition of placement p. D).

Then p. $D(0;0;\sqrt{2})$.

Draw from the vertex D of the bisector $DM$ i $DN$ flat corners $\angle ADC$ i $\angle ADB$ (Figure 2). By construction $MN$ – midline $\Delta$ABC, ie p. $M(x_3;y_3;0)$ – middle side AB, p. $N(x_4;y_4;0)$ – middle side AC.

Find the coordinates of the points $M$ i $N$:

$$x_3 = \frac{1+\left(-\frac{1}{2}\right)}{2} = \frac{1}{4};$$
$$y_3 = \frac{0+\frac{\sqrt{3}}{2}}{2} = \frac{\sqrt{3}}{4}; \quad \text{p. } M\left(\frac{1}{4};\frac{\sqrt{3}}{4};0\right),$$

$$x_4 = \frac{1+\left(-\frac{1}{2}\right)}{2} = \frac{1}{4};$$
$$y_4 = \frac{0-\frac{\sqrt{3}}{2}}{2} = -\frac{\sqrt{3}}{4}; \quad \text{p. } N\left(\frac{1}{4};-\frac{\sqrt{3}}{4};0\right).$$

Then, $\overrightarrow{DM} = \left\{\frac{1}{4}; \frac{\sqrt{3}}{4}; \sqrt{2}\right\}$; $\overrightarrow{DN} = \left\{\frac{1}{4}; -\frac{\sqrt{3}}{4}; \sqrt{2}\right\}$;

$$\cos \angle MDN = \frac{\overrightarrow{DM} \cdot \overrightarrow{DN}}{\left|\overrightarrow{DM}\right| \cdot \left|\overrightarrow{DN}\right|} = \frac{\frac{1}{4} \cdot \frac{1}{4} + \frac{\sqrt{3}}{4} \cdot \left(-\frac{\sqrt{3}}{4}\right) + \sqrt{2} \cdot \sqrt{2}}{\sqrt{\frac{1}{16} + \frac{3}{16} + 2} \cdot \sqrt{\frac{1}{16} + \frac{3}{16} + 2}} = \frac{5}{6},$$

where $\angle MDN = \arccos \frac{5}{6}$.

*The third method.*

Consider more general case of constructing a DABC tetrahedron with an arbitrary side $a$ and arbitrary coordinates of its vertices.

Suppose p. A $(x_1; y_1; z_1)$ according to the coordinates of this point we will consistently construct the other three points, indicating their coordinates.

We set Δ ABC in the XOY plane of the rectangular coordinate system. So, p. A $(x_1; y_1; 0)$, p. B $(x_2; y_2; 0)$, p. C $(x_3; y_3; 0)$.

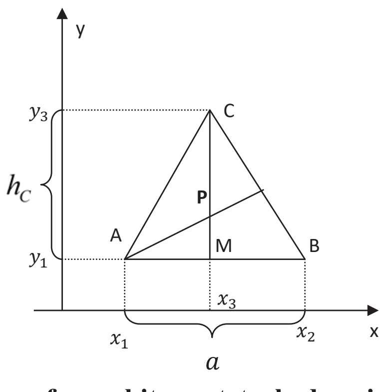


**Figure 4. Projection of an arbitrary tetrahedron into the XOY plane (side AB parallel to the OX axis)**

To simplify the calculations of the given model, let one of the sides (for example, the side AB) be parallel to one of the coordinate axes (for example, the OX axis) (Figure 4), then $\begin{cases} x_2 = x_1 + a \\ y_2 = y_1 \end{cases}$ p. B $(x_1 + a; y_1; 0)$.

Obviously, the coordinates of p. C Δ ABC will be as follows:

$$\begin{cases} x_3 = \dfrac{x_1 + x_2}{2}; \\ y_3 = y_1 + h_C; \end{cases}$$

$$\begin{cases} x_3 = \dfrac{2x_1 + a}{2}; \\ y_3 = y_1 + \dfrac{a\sqrt{3}}{2}; \end{cases} \quad \text{т. C}\left(\frac{2x_1 + a}{2}; y_1 + \frac{a\sqrt{3}}{2}; 0\right).$$

Suppose p. P is the point of intersection of the heights ΔABC, ie $CM = h_C = \frac{a\sqrt{3}}{2}$ on this PM=$\frac{1}{3}h_C = \frac{a\sqrt{3}}{6}$. Then, p. P$\left(\frac{2x_1 + a}{2}; y_1 + \frac{a\sqrt{3}}{6}; 0\right)$.

In turn, the fourth vertex D of the DABC tetrahedron is projected in p. P ΔABC and it will have the following coordinates: p. D$\left(\frac{2x_1 + a}{2}; y_1 + \frac{a\sqrt{3}}{6}; h_D\right)$, where $h_D = DP$ – the height of the tetrahedron DABC, dropped from the vertex D to the face Δ ABC.

With Δ DPB, we have: $BD^2 = BP^2 + PD^2$ or $\left|\overrightarrow{BD}\right|^2 = \left|\overrightarrow{BP}\right|^2 + \left|\overrightarrow{PD}\right|^2$, where $\left|\overrightarrow{PD}\right| = h_D$.

$$\overrightarrow{BP} = \left\{\frac{2x_1 + a}{2}; \frac{a\sqrt{3}}{6}; 0\right\}, \quad \left|\overrightarrow{BP}\right| = \sqrt{\frac{a^2}{4} + \frac{a^2}{12}} = \frac{a}{\sqrt{3}}.$$

Then, $a^2 = \frac{a^2}{3} + h_D^{\ 2} \Rightarrow h_D = \frac{a\sqrt{2}}{\sqrt{3}}$.

Therefore, т. D$\left(\frac{2x_1 + a}{2}; y_1 + \frac{a\sqrt{3}}{6}; \frac{a\sqrt{2}}{\sqrt{3}}\right)$.

Finding successively all four vertices of the tetrahedron DABC, each edge of which is equal $a$, find the coordinates of the points M and N, which correspond to the bases of the apophemes DM and DN of the constructed tetrahedron:

$$x_M = \frac{x_B + x_C}{2} = \frac{4x_1 + 3a}{4}; \quad \text{p. M}\left(\frac{4x_1 + 3a}{4}; \frac{4y_1 + a\sqrt{3}}{4}; 0\right).$$
$$y_M = \frac{y_B + y_C}{2} = \frac{4y_1 + a\sqrt{3}}{4};$$

$$x_N = \frac{x_B + x_A}{2} = \frac{2x_1 + a}{2}; \quad \text{p. N}\left(\frac{2x_1 + a}{2}; y_1; 0\right).$$
$$y_N = \frac{y_B + y_A}{2} = y_1;$$

Then, $\overrightarrow{DM} = \left\{\frac{a}{4}; \frac{a\sqrt{3}}{12}; -\frac{a\sqrt{2}}{\sqrt{3}}\right\}$ and $\overrightarrow{DN} = \left\{0; -\frac{a\sqrt{3}}{6}; -\frac{a\sqrt{2}}{\sqrt{3}}\right\}$,

$$\left|\overrightarrow{DM}\right| = \left|\overrightarrow{DN}\right| = \sqrt{\frac{3a^2}{36} + \frac{2a^2}{3}} = \frac{\sqrt{3}a}{2};$$

$$\overrightarrow{DM} \cdot \overrightarrow{DN} = -\frac{a^2}{24} + \frac{2a^2}{3} = \frac{15}{24}a^2$$ (scalar product of vectors).

Therefore, $\cos\angle MDN = \frac{\overrightarrow{DM} \cdot \overrightarrow{DN}}{\left|\overrightarrow{DM}\right| \cdot \left|\overrightarrow{DN}\right|} = \frac{\frac{15}{24}a^2}{\left(\sqrt{\frac{\sqrt{3}a}{2}}\right)^2} = \frac{5}{6}$, where $\angle MDN = \arccos\frac{5}{6}$.

The value of the trigonometric function of the required angle $\angle MDN$ can also be found using the vector product of some vectors. Since, for the area ΔDNM of the tetrahedron DABC the following relations are valid: $S_{\Delta DNM} = \frac{1}{2}\left|\overrightarrow{DM} \times \overrightarrow{DN}\right| = \frac{1}{2}\left|\overrightarrow{DM}\right|\left|\overrightarrow{DN}\right|\sin\angle\left(\overrightarrow{DM}; \overrightarrow{DN}\right)$, де $\overrightarrow{DM} \times \overrightarrow{DN}$ – vector product of vectors, whence $\sin\angle\left(\overrightarrow{DM}; \overrightarrow{DN}\right) = \frac{\left|\overrightarrow{DM} \times \overrightarrow{DN}\right|}{\left|\overrightarrow{DM}\right|\left|\overrightarrow{DN}\right|}$.

Whereas

$$\left|\overrightarrow{DM} \times \overrightarrow{DN}\right| = \left| \begin{vmatrix} i & j & k \\ \frac{a}{4} & \frac{a\sqrt{3}}{12} & -\frac{a\sqrt{2}}{\sqrt{3}} \\ 0 & -\frac{a\sqrt{3}}{6} & -\frac{a\sqrt{2}}{\sqrt{3}} \end{vmatrix} \right| = \left| -\frac{a^2\sqrt{2}}{4}i + \frac{a^2\sqrt{2}}{4\sqrt{3}}j - \frac{a^2\sqrt{3}}{24}k \right| = a^2\sqrt{\frac{1}{8} + \frac{1}{24} + \frac{1}{192}} = \frac{a^2\sqrt{11}}{8}$$

and $\left|\overrightarrow{DM}\right| = \left|\overrightarrow{DN}\right| = \dfrac{\sqrt{3}a}{2}$, then

$$\sin\angle\left(\overrightarrow{DM};\overrightarrow{DN}\right) = \frac{\frac{a^2\sqrt{11}}{8}}{\frac{3a^2}{4}} = \frac{\sqrt{11}}{6} \Rightarrow \cos\angle\left(\overrightarrow{DM};\overrightarrow{DN}\right) = \sqrt{1-\sin^2\angle\left(\overrightarrow{DM};\overrightarrow{DN}\right)} = \sqrt{1-\frac{11}{36}} = \frac{5}{6},$$

$$\angle MDN = \arccos\frac{5}{6}.$$

The fourth method.

Consider the correct tetrahedron DABC (Figure 5) and introduce the following notation: $\overrightarrow{AC} = \vec{a}, \overrightarrow{AB} = \vec{b}, \overrightarrow{AD} = \vec{c}$. It is known that $\angle\left(\vec{a};\vec{b}\right) = \angle\left(\vec{a};\vec{c}\right) = \angle\left(\vec{b};\vec{c}\right) = 60^\circ$ and $\left|\vec{a}\right| = \left|\vec{b}\right| = \left|\vec{c}\right| = 1$.

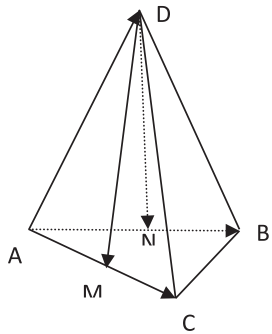


**Figure 5. The given tetrahedron in vectors**

By the property of vectors: $\overrightarrow{BC} = -\vec{b} + \vec{a}$ та $\overrightarrow{MN} = \dfrac{1}{2}\overrightarrow{BC} = \dfrac{1}{2}\left(\vec{a} - \vec{b}\right)$;

$\overrightarrow{DN} = -\dfrac{\vec{b}}{2} + \vec{c}; \overrightarrow{DM} = -\dfrac{\vec{a}}{2} + \vec{c}$. Then, $\cos\angle\left(\overrightarrow{DM};\overrightarrow{DN}\right) = \dfrac{\overrightarrow{DM}\cdot\overrightarrow{DN}}{\left|\overrightarrow{DM}\right|\left|\overrightarrow{DN}\right|}$, where

$$\overrightarrow{DN}\cdot\overrightarrow{DM} = \left(\vec{c} - \frac{\vec{b}}{2}\right)\cdot\left(\vec{c} - \frac{\vec{a}}{2}\right) = \left|\vec{c}\right|^2 - \frac{1}{2}\vec{a}\cdot\vec{c} - \frac{1}{2}\vec{b}\cdot\vec{c} + \frac{1}{4}\vec{a}\cdot\vec{b} = 1 - \frac{1}{2}\left|\vec{a}\right|\left|\vec{c}\right|\cos 60^\circ -$$

$$-\frac{1}{2}\left|\vec{b}\right|\left|\vec{c}\right|\cos 60^\circ + \frac{1}{4}\left|\vec{a}\right|\left|\vec{b}\right|\cos 60^\circ = 1 - \frac{1}{4} - \frac{1}{4} + \frac{1}{8} = \frac{5}{8};$$

$$\left|\overrightarrow{DN}\right|^2 = \left|\vec{c}\right|^2 - \left|\frac{\vec{b}}{2}\right|^2 = 1 - \frac{1}{4} = \frac{3}{4} \Rightarrow \left|\overrightarrow{DN}\right| = \left|\overrightarrow{DM}\right| = \frac{\sqrt{3}}{2}.$$

Therefore, $\cos\angle\left(\overrightarrow{DM};\overrightarrow{DN}\right) = \frac{\overrightarrow{DM}\cdot\overrightarrow{DN}}{\left|\overrightarrow{DM}\right|\cdot\left|\overrightarrow{DN}\right|} = \frac{\frac{5}{8}}{\frac{3}{4}} = \frac{5}{6}$, where $\angle MDN = \arccos\frac{5}{6}$.

Considering the methods of solving this problem, we understand how closely intertwined the variety of approaches to elementary and higher mathematics. Examples of some interesting methods and ways of solving a specific geometric problem that can be used in solving both standard and non-standard problems using both geometric and algebraic mathematical foundations are given. Such research provides an opportunity to expand mathematical knowledge, demonstrate that mathematics is a living and interesting science.

### 3. Calculation of work in one applied technical problem with different geometry of arrangement and comparison of the received results

There is a cylindrical tank with the appropriate geometric parameters: *R*-the radius of the circle of the base of this three-dimensional figure, *H*-its height. This container is filled with some liquid density *ρ*. The main task of these studies is comparative numerical characteristics of the pumping (raising to the surface of the tank) of the volume of this liquid in two cases: in the first – if the geometric structure has a horizontal arrangement of its bases (Figure 6), in the second – when such bases of the cylindrical structure will have a vertical position (Figure 7).

**Research results.** In determining the magnitude of such work in each of these cases the well-known physical law [4-6] should be used, according to which in the first case (Figure 6) to raise elementary mass of liquid mass to the surface of such a reservoir $\Delta m_1 \approx \pi\rho R^2 dy$, located at an arbitrary height *y*, the magnitude of the elementary work $\varnothing A_1$ is roughly defined as the product of the elementary force of gravity $\Delta F \approx g\Delta m$ to the appropriate lifting height equal to $(H-y)$, herewith $y\in[0;H]$, thus

$$\Delta A_1 \approx \pi\rho g R^2 (H-y)\ \Delta y.$$

As can be seen from the last formula, the value of the elementary work on raising the elementary volume of the liquid allotted at arbitrarily taken y

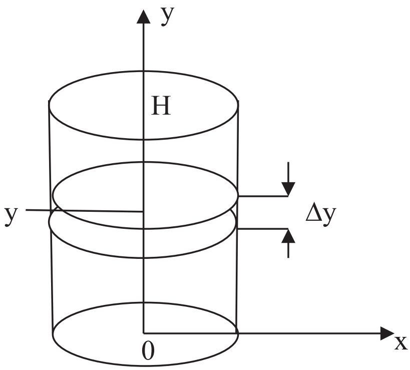


**Figure 6. Cylindrical tank filled with liquid and horizontally arranged bases**

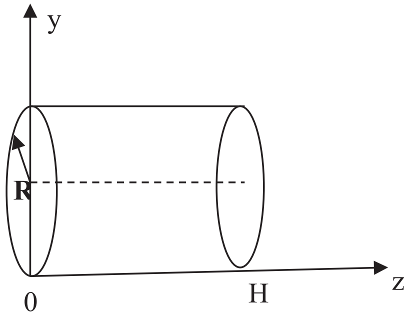


**Figure 7. Cylindrical tank filled with liquid and vertically arranged bases**

at the appropriate height for raising to the surface of the tank is the value of differentiated (variable) depending on the same ordinate.

If in this equation provided that $\Delta y$ goes to zero, $\varnothing A_1$ will also go to zero, therefore, the corresponding increments of the argument and its dependent function can be replaced by differentials of these infinitesimal quantities, and the approximate equal sign can be replaced by an exact one, ie $d\,A_1 = \pi\rho g R^2 \left(H-y\right) d$ y.

Whereas $A_1 = \int\limits_0^H dA_1\left(y\right)$,

then, having integrated the right-hand side of the last equality on the variable $y$ within the possible change of this argument, replacing the approximate sign of equality with the exact one, we have the final result of the desired work $A_1$:

$$A_1 = \int\limits_0^H \pi\rho g R^2 \left(H-y\right) dy = -\frac{1}{2}\int\limits_0^H \pi\rho g R^2 d\left(H-y\right) = -\frac{1}{2}\pi\rho g R^2 \left(H-y\right)^2 \Big|_0^H = \frac{1}{2}\pi\rho g R^2 H^2 .$$

Similarly, set the amount of work $A_2$ required to raise the entire volume of liquid to the surface of the tank, when the bases of such a cylindrical structure will be located vertically (Figure 7). The equation of the arc of the base circle in the selected coordinate system is defined as $x^2 + (y-R)^2 = R^2$, where $x = \pm\sqrt{R^2 - (y-R)^2}, y \in [0;2R]$. At an arbitrary height *y*, the elementary mass of the fluid layer at such an arrangement of a given cylindrical structure will be determined as follows:

$\Delta m_2 \approx 2x\rho H \Delta y$, based on the appropriate formula for determining the volume of an elementary parallelepiped of height $\Delta y$. Then the elementary force required to raise to the surface of such a structure all the particles of the liquid from the specified height of the location of such particles in and taking into account the given value of $x$ will be defined as

$$\Delta F_2 \approx 2\rho g H \sqrt{R^2 - (y-R)^2}\,\Delta y, y \in [0;2R].$$

So, this elemental force is equal to the force of gravity emitted at the height of the elementary volume for a given fluid. Further, given the fact that the particles of this elementary volume of liquid must be raised to the height of rise $(2R - y)$, the element of work $\Delta A_2$ will be defined as

$$\Delta A_2 \approx 2\rho g H \sqrt{R^2 - (y-R)^2}\,(2R - y)\Delta y, y \in [0;2R].$$

As in the first case, replacing the increments of the function and its argument by the corresponding differentials of these infinitesimal quantities, we have a definite expression in the form of a definite integral for the final establishment of the value of the total sum $A_2$ as follows:

$$\mathrm{A}_2 = \int_0^{2\mathrm{R}} 2\rho g H (2R - y)\sqrt{R^2 - (y-R)^2}\,dy = \begin{pmatrix} R - y = t \\ dy = -dt \\ t \in [-R;R] \end{pmatrix} =$$

$$= 2\rho g H \int_{\mathrm{R}}^{-\mathrm{R}} (R - t)\sqrt{R^2 - t^2}\,d(-t) =$$

$$= 2\rho g H R \int_{-\mathrm{R}}^{\mathrm{R}} \sqrt{R^2 - t^2}\,dt - 2\rho g H \int_{-\mathrm{R}}^{\mathrm{R}} \mathrm{t}\sqrt{R^2 - t^2}\,dt = 2\rho g H R \int_{-R}^{R} \sqrt{R^2 - t^2}\,dt =$$

$$= 4\rho g H R \int_0^R \sqrt{R^2 - t^2}\,dt = 4\rho g H R \frac{\mathrm{t}\sqrt{R^2 - t^2}}{2}\Big|_0^R + 4\rho g H R \frac{R^2}{2} \arcsin\frac{t}{R}\Big|_0^R = \pi\rho g R^3 H.$$

Here $\frac{t\sqrt{R^2-t^2}}{2}\Big|_0^R = 0$, also $arcsin 1 = \frac{\pi}{2}$, $\int_{-R}^{R} t\sqrt{R^2-t^2}dt = 0$ as an integral of the odd function under the sign of the integral symmetric with respect to the point 0 of the integration interval.

Thus:

$$A_1 = \frac{1}{2}\pi\rho g R^2 H^2,$$

$$A_2 = \pi\rho g R^3 H.$$

Based on the latest results, these values can be compared and their corresponding ratio is taken:

$$\frac{A_1}{A_2} = \frac{H}{2R}, \text{ або } A_1 = \frac{H}{2R} A_2 .$$

The following features are obtained:

$$\frac{H}{2R} > 1 \Leftrightarrow H > 2R \Leftrightarrow A_1 > A_2,$$

$$\frac{H}{2R} < 1 \Leftrightarrow H < 2R \Leftrightarrow A_1 < A_2,$$

$$\frac{H}{2R} = 1 \Leftrightarrow H = 2R \Leftrightarrow A_1 = A_2 .$$

Thus, the values of the required works in both cases will be equal when the height (generative) H of such a cylindrical structure coincides with its diameter 2*R*, which is the logical result of such studies.

It should also be noted that if the position of the cylindrical structure is horizontal at its base, the magnitude of the corresponding work will be proportional to both the square of the radius and the square of the height of such structure (Figure 6), and if such capacity is horizontal with vertical both bases (Figure 7), the magnitude of the desired work will already be proportional to both the cube of the radius of the base circle and the height (creative) of such a structure of the corresponding geometry.

## 4. Setting the maximum number of filling of some geometric structure with many other geometric figures of circular (spherical) shape

Questions about the optimal location of certain geometric objects (structures) are among the problems of optimal geometric design. Such studies are relevant and their results are constantly growing both from a purely theoretical and practical point of view [7–12]. In the studies, the so-called coefficient of useful filling of one geometric structure with many other objects plays a very important role. As a rule, the main task of such researches is to increase the numerical value of the coefficient and its direct estimation in one or another mathematical model that is being studied [11–12]. The results of them have a wide range of possible applications: the problem of maximum filling of tanks of different geometric shapes in the agricultural sector of the economy, mechanical engineering, medicine, in particular, pharmacology, light, furniture, etc. The basis of such problems and their solution is to determine the optimal location of a finite set of certain geometric objects within certain geometric structures. We aim to calculate the possibilities of maximum filling of a given geometric figure (both flat and three-dimensional) by a set of circles (respectively, balls) of the same radius $r$, namely, the quantitative component of such optimal filling, and formulate some criteria for estimating such maximum filling. Obviously, such a criterion can be chosen as the ratio of the total usable area (volume) of the filling of a given geometric structure to the area (volume) of this given geometric structure.

The purpose of the researches is to establish the optimal filling of a certain external geometric structure with a finite set of objects of one or another geometric shape, to establish the quantitative characteristics of such filling, to introduce and calculate the coefficient of useful maximum filling of such structure.

In order for the external geometric structure to reach its maximum content, we will require that the linear dimensions of a given geometric structure be multiples of the numerical value $d=2r$ (d-diameter). Under this necessary condition, the area (volume) of unproductive voids of the external geometric structure is minimized and, accordingly, the filling of such a structure will reach the highest values.

1. If in the plane case (ie in two-dimensional space) as an external geometric structure is chosen a rectangle with the dimensions of the

sides, respectively, *a* and *b*, whose numerical values are multiples of 2*r*, ie $\frac{a}{2r}=m, \frac{b}{2r}=n$, where $m,n-$ integers, then the quantitative value of the optimal (maximum) filling of such a flat geometric structure is easily established and is equal to:

$N=mn$.

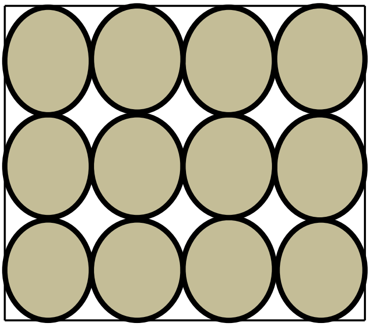

**Figure 8. Rectangular structure, where *m* = 4, *n* = 3**

As an example of such a simpler structure, Figure 8 shows a structure in the form of a rectangle with values of $m=4, n=3$. Then $N=12$. Now in this example the degree (coefficient) of efficiency of such filling of the set structure as the relation of the total area of all 12 circles to the area of the set external rectangular structure will be established. Here:

$$\xi=\frac{mn\pi r^2}{ab}100\%=\frac{mn\pi r^2}{4mnr^2}100\%=\frac{\pi}{4}100\%=78,5\%.$$

An analogue of the above structure, but with the transition to three-dimensional space is a straight parallelepiped, which is filled as much as possible by a set of balls of fixed radius, while the linear dimensions of the independent sides of the parallelepiped meet the conditions:

$\frac{a}{m}=\frac{b}{n}=\frac{c}{k}=2r$, where $m,n,k-$ integers, $2r-$ diameter of each sphere.

In this case, the maximum possible number of such balls inside the specified parallelepiped will be equal to:

$N=mnk$, the coefficient of useful effective filling of such a structure of three-dimensional space will be determined as the ratio of the total useful volume to the volume of the outer parallelepiped. Summarizing the example shown in Figure 8 on three-dimensional space and assuming, for example, $k$ = 3, the value of this coefficient will be as follows:

$$\xi = \frac{\frac{4}{3}\pi r^3 mkn}{abc}100\% = \frac{\frac{4}{3}\pi r^3 mkn}{8mnkr^3}100\% = \frac{\pi}{6}100\% = 52{,}3\%.$$

From the results of finding the coefficients of optimal filling of similar flat and three-dimensional models, we can conclude that the optimum maximum filling for a flat model is 1.5 times higher than in a similar three-dimensional model. The next conclusion is that the numerical values of the corresponding coefficients of useful filling of flat and three-dimensional structures do not depend on the linear dimensions of the outermost geometric structure.

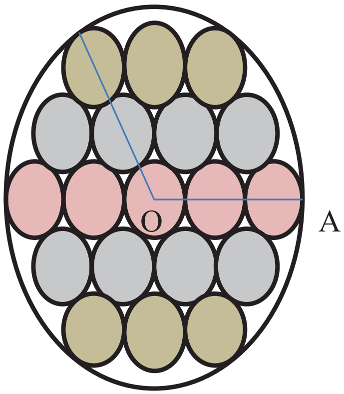


Figure 9. The example of a flat outer circular structure of radius $R$, maximally filled with a set of circles of fixed radius $r, R > r$ in the case when $\frac{R}{r}-$ is an odd number ($\frac{R}{r} = 5$).

2. As another external geometric structure in two-dimensional space, we introduce a circle of radius $R, (R > r)$, here $r$ – values of the fixed radius of the maximum possible number of inner circles, which fill the area of the outer structure of a large circle. We assume that $\frac{R}{r} = l$, де $l-$ is an integer.

According to the research, this point can be further divided into two sub-items depending on the parity – the oddness of the numerical value of *l*.

A) Suppose $l = 2k-1, k-$ integer. The case of the so-called paired model. An example of such a specific structure $k = 3(l = 5)$ is shown in Figure 9.

The dependence of the maximum possible number of small circles inside the outer circular structure has been established and this dependence is determined by the formula of the following form:

$$N\left(R=(2k-1)r\right)=2k-1+2\sum_{i=1}^{k-1}(2k-1-i), k=2,3,4\dots$$

This formula at large values of *k* is not particularly convenient, so we offer a condensed version, which is more appropriate, it has the following form:

$$N\left(R=(2k-1)r\right)=3k^2-3k+1, k=2,3,4\dots$$

According to the latter result, the value of the number *N* increases in quadratic with respect to *k* dependence.

Table 1

**Dependence of the maximum possible number of small circles on the value of the integer *k*, which are located inside the large circle**

| k | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| The number of circles N of radius r depending on the value k (R= (2k-1)r) | 7 | 19 | 37 | 61 | 91 |

Using the above result, we can easily set the maximum possible number of small circles to fill a large circle in Figure 9: $N(R=5r)=27-9+1=19$.

The established dependence of the number *N* on the value of *k* can also be set using table 1.

B) Suppose $l = 2k, k-$ integer. This is the case with the so-called paired model. An example of such a specific structure $k = 3(l = 6)$ is shown in Figure 10.

In this case, the dependence of the maximum possible number of small circles inside the outer circular structure is determined by the formula of the form:

$$\mathrm{N}\left(\mathrm{R}=2\mathrm{kr}\right)=2\mathrm{k}+2\sum_{\mathrm{i}=1}^{\mathrm{k}-1}(2\mathrm{k}-\mathrm{i}), \mathrm{k}=2,3,4,\dots$$

In this case, the dependence of the maximum possible number of small circles inside the outer circular structure is determined by the formula of the form:

$$N(R=2kr)=2k+2\sum_{i=1}^{k-1}(2k-i),k=2,3,4,\dots$$

Based on this result, we have a similar quadratic relationship between the values *N* and *k*.

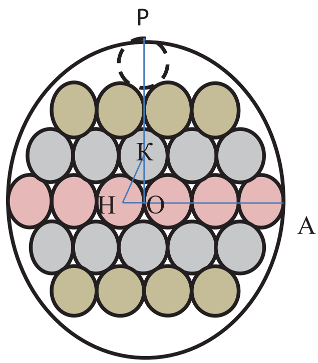


Figure 10. An example of a flat outer circular structure of radius *R*, maximally filled with a set of circles of fixed radius $r, R > r$, in the case where $\frac{R}{r}$ – is an even number ($\frac{R}{r}=6$).

As a remark to sub-item B) it is noted that visually it is possible to get an impression of the placement of several additional circles, one of which is possible in Figure 10 is dotted. But research has shown that such additional placement is impossible because: $OA=6r, OP=3OK+OH=3\sqrt{3}r+r=(3\sqrt{3}+1)r\approx 6,1r>6r$. The established dependence can be similarly interpreted in the form of table 2.

Table 2

**Dependence of the maximum possible number of small circles on the value of the integer *k*, which are located inside the large circle**

| k | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| The number of circles N of radius r depending on the value of k (R = 2kr) | 10 | 24 | 44 | 70 | 102 |

The execution of the last inequality mathematically strictly proves the fact that the presence in Figure 10 of the dotted additional possible circle is actually impossible.

Let us dwell on the question of establishing the value of the coefficient of effective (maximum) filling of such an external circular structure. For the odd model we have the following estimate for this coefficient:

$$\xi_{неп} = \xi\left(R = (2k-1)r\right) = \frac{N\left(R = (2k-1)r\right)\pi r^2}{\pi R^2}100\% = \frac{N\left(R = (2k-1)r\right)\pi r^2}{\pi r^2 (2k-1)^2}100\% = \frac{3k^2 - 3k + 1}{(2k-1)^2}100\% \approx 75\% .$$

It is similarly for the pair model:

$$\xi_{п} = \xi\left(R = 2kr\right) = \frac{N\left(R = 2kr\right)\pi r^2}{\pi R^2}100\% = \frac{N\left(R = 2kr\right)\pi r^2}{\pi r^2 (2k)^2}100\% = \frac{3k^2 - k}{(2k-1)^2}100\% \approx 75\% .$$

As we can see, the asymptotic values of the given coefficients coincide essentially with one constant value equal to 75%.

Then, based on the principles of calculating the outer circular structure, we can further introduce a cylindrical outer geometric structure with the parameters R – radius of the base circle and H – height in three-dimensional space. In this case, we consider the value of H – to be a multiple of 2*r*, where *r* – is the fixed radius of each of the balls, which can fill the maximum volume of the cylinder. If $\frac{H}{2r} = p$, where *p* – is an integer, then in relation to the maximum possible number of small balls that can be placed inside the cylinder, we obtain:

$$Q = Np.$$

In this case, the coefficient of maximum useful filling of such a three-dimensional structure is defined as

$$\xi = \frac{\frac{4}{3}\pi r^3 Q}{\pi R^2 H}100\% = \frac{4}{3}\frac{r^3 Np}{(2k-1)^2 r^2 2pr}100\% = \frac{2}{3}\frac{3k^2 - 3k + 1}{(2k-1)^2}100\% \approx 50\% .$$

Thus, we have a similar similarity with respect to the considered external circular structure, namely, the coefficient of volumetric maximum filling is 1.5 times smaller in comparison with the same coefficient of flat filling of the corresponding circular structure.

As follows:

– cases of maximum filling of flat and three-dimensional geometric structures with a finite set of geometric objects are considered, the quantitative measure of each of the given fillings is established;

– the coefficient of such useful filling of the corresponding geometrical structure both in two-dimensional and three-dimensional spaces is entered, the estimation of calculation of such coefficient is made;

– a comparative analysis of such coefficients for flat and three-dimensional cases of solving the problem posed in the research.

## 5. Conclusions

The main results of the research of this work can be considered the following:

– in the first part of this work, on the example of solving one geometric problem, mathematical versatility of different approaches and methods of solving this particular problem and many other mathematical problems is established, therefore, the question in such cases is not just to find the right solution to the problem to be solved, but also to choose the solution that would be the most acceptable and optimal;

– the second part investigates on the basis of known classical results of one of the physical laws comparative analysis of one of the numerical characteristics, namely, work in this case using the mathematical apparatus of integral calculus, established the relationship of relevant models the body to which the essence of this task is attached;

– in the third part the problem of maximum filling of certain external spatial geometric figures with a set of homogeneous circular or spherical objects is solved, the so-called coefficient of such useful filling is offered and tested.

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