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MATHEMATICAL MODEL AND SOLUTION OF SPATIAL CONTACT PROBLEM FOR PRESTRESSED CYLINDRICAL PUNCH AND ELASTIC LAYER

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INTRODUCTION

Mathematical modeling and research of problems of contact interaction of pre-stressed solids is quite relevant in our time. Confirmation of this, is the speech titled «On the implementation of the target program of scientific research of the NAS of Ukraine» Reliability and durability of materials, constructions, equipment and structures»« by Leonid Lobanov, the Academician of the NAS of Ukraine which took place on December 09, 2020[1]. Considering this fact, it is important to solve the problems of contact interaction of deformed solids dealt with load transfer in constructions, structures and parts of machines, which related with the presence of initial stresses in solids according to the law of division of contact stresses and displacements.

Contact problems are an important part of the mechanics of a deformable solid and form the theoretical basis for calculations for the contact strength, stiffness and wear resistance of mobile and fixed joints.

The applied needs of natural science, modern technology and the latest technologies in recent decades associated with the necessity to predict the contact behavior of various designs, stimulated the development of various mathematical models and methods of contact mechanics of bodies with different properties[2].

One of the important factors in the contact interaction of bodies is the influence of initial (residual) stresses. Despite a significant achievement in the development of contact problems, nevertheless the issue of taking into account the initial (residual) stresses in the contact interaction has remained almost completely undeveloped until recently. There is known,

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[1] Повідомлення НАН України. URL: http://www.nas.gov.ua/UA/Messages/Pages/View.aspx?MessageID=7263

[2] Babich. S. Yu., Dikhtyaruk N. N. Load transfer from an infinite inhomogeneous stringer to an elastic strip clamped by one face with initial stresses. *I*nternational Applied Mechanics. 2020. Vol. 56. №6. P. 346–356. https://doi.org/10.1007/s10778-021-01047-9

that almost all elements of the construction have initial stress. It can be caused by various reasons, for example, by technological operations conducted in the manufacture of a variety of materials or by assembly of a structure. In the case of composite materials, the initial stresses, as a rule, correspond to stresses along the reinforcing elements. In the earth's crust, they are formed due to the action of gravitational forces and technical processes. They must be taken into account when solving the problems of deformation of soils (especially frozen ones). In addition, in elastoplastic bodies, internal residual stresses can also be present after removal of loads.

In the general case, consideration of the initial (residual) stresses requires the using of the apparatus of the nonlinear theory of elasticity[3], but for the sufficiently large initial (residual) loads, one can confine ourselves to its linearized version[4].

Therefore, this article offers the mathematical model and the solution of the contact problem about the pressure of the pre-stressed cylindrical punch to the elastic layer with initial stresses. The study of the problem carried out within the linearized theory of elasticity[5] without taking into account the forces of friction.

1. The problem's prerequisites emergence and review of literature sources

Linearized theory of elasticity for the bodies with initial (residual) stresses as the linearization of the nonlinear theory of elasticity[6] was first proposed in monograph[7]. Also, in his work[8], the author, using considerations of a physical nature and not always strictly adhering to the principle of linearization of the nonlinear theory, developed the theory of

[3] *Guz A. N.* Nonclassical Problems of Fracture/Failure Mechanics: On the Occasion of the 50th Anniversary of Research (Review). III. *I*nternational Applied Mechanics. 2019. Vol. 55. №4. Pp. 343–415. https://doi.org/10.1007/s10778-019-00960-4

[4] Гузь А.Н., Бабич С.Ю., Глухов Ю.П. Смешанные задачи для упругого основания с начальными напряжениями. Германия: Saarbrücken LAPLAMBERT Academic Publishing, 2015. 468 с

[5] Гузь А.Н., Рудницкий В.Б. Основы теории контактного взаимодействия упругих тел с начальными (остаточными) напряжениями. Хмельницький: вид. ПП Мельник, 2006. 710 с.

[6] Rushchitsky J.J. Auxetic metamaterials from the position of mechanics: linear and nonlinear models. *Dopov. Nac. akad. nauk Ukr*. 2018. №7. P. 46 – 58. doi: https://doi.org/10.15407/dopovidi2018.07.046.

[7] Kappus R. Zur Elastizctatstheorie tndlicher Verschiebunger. *ZAMM*. 1939. Vol. 19. №5. P. 271–15.

[8] Biot M. A. Mechanics of incremental deformations. New York: John Willey and Sons. 1965. 126 p.

incremental deformations for the bodies with initial stresses. Nevertheless, a simplified version of such theory based on a physical nature was considered by Cauchy (XIX century). Today the results of the works[9] are completely based on the linearized theory of elastic bodies with initial (residual) stresses.

The fundamental results of the linearized theory of elasticity were obtained by academician Gusem A.N.[10]. For the first time, he solved a number of contact problems for compressible and incompressible bodies by one of the most effective approaches for materials with an arbitrary form of elastic potential and homogeneous initial (residual) stresses. This approach is based on the theory of the function of a complex variable for plane problems and potential theory for spatial problems. Further development of the theory of contact interaction[11] of bodies with initial (residual) stresses was obtained in the works[12]. A general analysis of the main methods and the best known results in all directions of the contact interaction of bodies with initial (residual) stresses is presented in review articles[13].

Allowance of initial (residual) stresses within the linearized theory of elasticity leads to a new formulation of the problems of interaction of deformable solids, which significantly differ from the formulation of the problems of the classical theory of elasticity. Taken into account the problems when the initial (residual) stresses of the system of basic differential equations, the expressions of determining the components of the tensors of the stress-strain state and the structure of the boundary conditions differ from the corresponding systems of equations and expressions of the classical theory of elasticity, nevertheless, in their structure and nature they are similar to ordinary contact problems. Thus,

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[9] Гузь О.М., Бабич С.Ю., Рудницький В.Б. Контактна взаємодія тіл з початковими напруженнями: Навчальний посібник. Київ: Вища школа, 1995. 304 с.

[10] Guz, O.M., Babych, S.Y., Glukhov, A.Y. Axisymmetric Waves in Prestressed Highly Elastic Composite Material. Long Wave Approximation. *International Applied Mechanics*. 2021. Vol. 57. №2. P. 134–147. https://doi.org/10.1007/s10778-021-01068-4

[11] Babych, S.Y., Glukhov, Y.P. On One Dynamic Problem for a Multilayer Half-Space with Initial Stresses. *International Applied Mechanics*. 2021. Vol. 57. № 1. P. 43–52. https://doi.org/10.1007/s10778-021-01061-x

[12] Yaretskaya N. A. Three-Dimensional Contact Problem for an Elastic Layer and a Cylindrical Punch with Prestresses. *I*nternational Applied Mechanics. 2014. Vol. 50, № 4. P. 378–388. https://doi.org/10.1007/s10778-014-0641-y

[13] Гузь А. Н., Бабич С.Ю., Рудницкий В.Б. Контактное взаимодействие упругих тел с начальными (остаточными) напряжениями. *Развитие идей Л. А. Галина в механике*. М.: Ижевск. Институт компьютерных исследований. 2013. 480 с.

from the above, it follows the possibility of using many fundamental results and methods of the linear theory of elasticity.

The first equations of the linearized theory of elasticity of deformable bodies[14] were obtained by linearizing the basic relations of the nonlinear theory, taking into account the physical characteristics of the materials; these results are obtained for small subcritical deformations in Lagrangian coordinates, which coincide with the Cartesian coordinates in the undeformed state. Later the main relations were written in curvilinear coordinates using the tensor analysis[15]; equations in displacements were also obtained, for which in a homogeneous subcritical state some methods for their solution are considered.

A modern analysis of the approaches to constructing theories and basic results that are applied to the three-dimensional linearized theory of elasticity of deformable bodies and the three-dimensional linearized theory of the propagation of elastic waves in bodies with initial (residual) stresses is presented by the generalizing publication, respectively[16]. With the using of approaches of the modern type[17], modern analysis of the results is performed for a wide range of problems of the linearized mechanics of deformed bodies, namely:

1) For problems of the contact interaction of elastic bodies with initial (residual) stresses[18];

2) For the stability theory of the local equilibrium state of black rocks[19];

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[14] Guz A. N., Guz I.A The stability of the interface between two bodies compressed along interface cracks. 3. Exact solutions for the combined case of equal and unequal roots. *Int. Appl. Mech*. 2000. Vol. 36. №6. P. 759 – 768. https://doi.org/10.1007/BF02681983.

[15] Neuber H. Theorie der elastischen Stabilitat bei nichtlinearer Vorverfomung. *Acta mech*. 1965. Vol. 1. №3. P. 112-32

[16] Guz A.N. Elastic waves in bodies with initial (residual) stresses. *Int. Appl: Mech*. 2002. Vol. 38. № 1. P. 23–59. https://doi.org/10.1023/A:1015379824503

[17] Гузь А.Н., Бабич С.Ю., Глухов Ю.П. Статика и динамика упругих оснований с начальными (остаточными) напряжениями: Монография. Кременчук: «Press – Line», 2007. 795 с.

[18] Guz A.N., Babich S.Y., Rudnitskii V.B. Contact problems for elastic bodies with initial stresses: Focus on Ukrainian research. Int. Appl. Mech. Rew. 1998. Vol. 51. № 5. P. 343–371. https://doi.org/10.1115/1.3099009

[19] Guz A. N., Guz I.A The stability of the interface between two bodies compressed along interface cracks. 3. Exact solutions for the combined case of equal and unequal roots. *Int. Appl. Mech*. 2000. Vol. 36. № 6. P. 759–768. https://doi.org/10.1007/BF02681983

3) For exact solutions of plane mixed problems of linearized mechanics of deformable bodies[20];

4) For non-destructive ultrasonic methods for determining stresses in solids[21].

There are also a number of other generalizing publications about linearized mechanics. Moreover, the works mentioned above are only fully or partially related to the subject matter of this article. More widely the history of development and the range of problems of the linearized theory of elasticity are presented in[22].

Thus, the development of effective methods for calculating the stress-strain state with allowance for the initial (residual) deformations within the framework of the linearized theory of elasticity is an actual and important scientific and technical problem.

Today, in accordance to the problems related to contact problems for elastic bodies, results have been obtained on a wide range of issues. They are represented by works[23]. There are also a number of general publications[24], which are fully or partially related to the subject of this study. Despite significant achievements, the number of studies on the contact interaction of prestressed bodies is relatively small.

A rather detailed review of the work of rigid stamps (including ring ones) associated with contact pressure in the case of absence of initial stresses is given in the monograph[25].

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[20] Guz A. N., Guz I.A. Mixed plane problems in lianearized solid mechanics. Exact solutions. *Int. Appl. Mech*. 2004; Vol. 40. № 1. С. 1–29. https://doi.org/10.1023/B:INAM.0000023808.08859.48

[21] Guz A. N. Establishing the fundamentals of the theory of stability of mine workings. *Int. Appl. Mech*. 2003. Vol. 39. № 1. P. 20–48. https://doi.org/10.1023/A:1023659931802.

[22] Гузь А. Н., Бабич С.Ю., Рудницкий В.Б. Контактное взаимодействие упругих тел с начальными (остаточными) напряжениями. *Развитие идей Л. А. Галина в механике*. М.: Ижевск. Институт компьютерных исследований. 2013. 480 с.

[23] Babych, S.Y., Yarets'ka, N.O. Contact Problem for an Elastic Ring Punch and a Half-Space with Initial (Residual) Stresses. Int Appl Mech. 2021. Vol. 57. № 3. P. 297-305. https://doi.org/10.1007/s10778-021-01081-7.

[24] Guz A.N., Babich S.Y., Rudnitskii V.B. Contact problems for elastic bodies with initial stresses: Focus on Ukrainian research. Int. Appl. Mech. Rew. 1998. Vol. 51. № 5. P. 343–371. https://doi.org/10.1115/1.3099009

[25] Развитие теории контактных задач в СССР / под ред. Л. А. Галина. М.: Наука, 1976. 494 с.

The contact interaction of rigid and elastic stamps with prestressed bodies is presented in[26]. Moreover, either the elastic potentials of a particular structure are considered, but also the problem is considered in a general form for compressible (incompressible) bodies with the potential of an arbitrary structure on the basis of the linearized theory of elasticity.

The influence of initial stresses on the contact interaction of a rigid ring stamp on an elastic half-space with initial (residual) stresses is presented in[27].

In this paper, the investigation is carried out in a general form for compressible and incompressible bodies for the theory of large initial deformations and for two versions of the theory of small initial deformations for an arbitrary structure of the elastic potential.

2. Statement of the problem, basic relationships and border conditions

In the study of the contact interaction of solids with initial stresses, we will distinguish the following states of the body: 1) non-deformable (natural) – there are no deformations and stresses; 2) deformable (initial, main) – there are initial (residual) deformations and stresses; 3) state of perturbation. All values that will relate to the deformation state as in [28], will be marked with the upper index "0", and the values of the perturbed state as in [29] – stroke. The second and third states are the equilibrium states of the solid or its movement. They can be described using the nonlinear theory of elasticity of finite, first, and second variants of small initial deformations. Moreover, for the first variant of the nonlinear theory of small initial deformations, assumed that the relations between the elongations and shifts can be neglected because they are smaller than one. For the second variant of the nonlinear theory of small initial deformations, except the assumptions of the first variant, is added the assumption that the deformable state of the solid can be determined by the

[26] Aleksandrov V. M. , Arutyunyan N. Ky. Contact problems for prestressed deformed bodies. *Soviet Applied Mechanics*. 1984. Vol. 20. №3. P. 209–215. https://doi.org/10.1007/BF00883134

[27] Yaretskaya N. A. Contact Problem for the Rigid Ring Stamp and the Half-Space with Initial (Residual) Stresses. *I*nternational Applied Mechanics. 2018. Vol. 54, № 5. P. 539-543. https://doi.org/10.1007/s10778-018-0906-y

[28] Гузь А.Н., Рудницкий В.Б. Основы теории контактного взаимодействия упругих тел с начальными (остаточными) напряжениями. Хмельницький: вид. ПП Мельник, 2006. 710 с.

[29] Гузь А.Н., Рудницкий В.Б. Основы теории контактного взаимодействия упругих тел с начальными (остаточными) напряжениями. Хмельницький: вид. ПП Мельник, 2006. 710 с.

geometric linear theory, and in comparison with the one, it is possible to neglect the components of the vector of displacement of points of the solid in the initial state, that is, $\delta_{ij} + \partial U_j^0 / \partial x_i \approx \delta_{ij}$, where δ_{ij} – components of a metric tensor in a non-deformable state.

In addition, the quantities of the third state will be described as the sum of the values of the deformable state and their corresponding perturbations, which will be considered smaller than the values of the second state.

The ratio for the third state is called the ratio of linearized elasticity theory for solids with initial (residual) stresses, when their linearization to subtract quantities that correspond to the deformable state of the solid. That is, if $z=f(x)$ – some ratio of nonlinear elasticity theory, so the ratio of linearized theory will look as $z \approx x\,(df/dx)\big|_{x=x_0}$ [30].

For the study, we use the coordinates of the initial deformed state (y_1, y_2, y_3), which are related to the Lagrange coordinates (x_1, x_2, x_3) (natural state): $y_i=\lambda_i x_i$ $(i=\overline{1,3})$. Here λ_i $(i=\overline{1,3})$ are the elongation coefficients that determine the movement of the initial state $\lambda_i = \text{const}$ $(i=\overline{1,3})$. The y_3 axis is directed along the normal to the contact area.

Suppose that the initial states of the contact solids are homogeneous and equal, and elastic potentials are twice continuously differentiated functions of algebraic invariants of the Green deformation tensor[31]. Materials of solids, that we consider isotropic compressive or incompressible with an arbitrary structure of elastic potential.

All quantities related to the elastic cylinder are denoted by the superscript "(1)," the layer – "(2)," and the bases – "(3)."

In the study that we're considering elastic isotropic solids (compressive or incompressible) with an arbitrary form of elastic potential. In the case of orthotropic solids, we assume that elastic-equivalent directions coincide with the direction of coordinate axes in a deformable state y_i $(i=\overline{1,3})$. Let the initial deformable state be homogeneous and the contact area of the elastic solids be contained in the y_3=const plane. Let's assume that the initial stresses operate along the contact zone.

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[30] Гузь А.Н., Рудницкий В.Б. Основы теории контактного взаимодействия упругих тел с начальными (остаточными) напряжениями. Хмельницький: вид. ПП Мельник, 2006. 710 с.

[31] Гузь А.Н., Рудницкий В.Б. Основы теории контактного взаимодействия упругих тел с начальными (остаточными) напряжениями. Хмельницький: вид. ПП Мельник, 2006. 710 с.

Since the initial stresses are not taken into account in linear mechanics of materials, it is possible to apply the general nonlinear theory of elasticity [32]. In this case, it will be quite difficult to get the solution in the available form. Therefore, with the significant amount of initial stresses, it is better to use its linearized option.

Thus, suppose that the following provisions[33], will always be met, which are fundamental to linearized elasticity theory:

1. The contact interaction of an elastic finite cylindrical die with initial stresses with a pre-stressed elastic layer occurs after the initial stressed state.

2. An additional external load (relative to the initial state) acts on an elastic cylindrical stamp, causing much less disturbance of the stress-deforming state in a layer with initial (residual) stresses compared to the corresponding values of the initial stressed state.

3. The initial stress-strain state of the bodies of contact interaction has such a structure that in the area of their contact it can be approximately considered homogeneous.

4. The solution of the linearized problem of elasticity theory on the contact interaction of a pre-stressed cylindrical die with an elastic layer with initial stresses is the only one, that is, the condition is met[34].

The above provisions make it possible to apply linearized elasticity theory to solve this problem. Note that in particular the second position may be violated in the vicinity of the points of change of boundary conditions, in which contact stresses rise to infinity. Detailed discussion of this phenomenon in the theory of contact problems of linear and linearized elasticity theory is performed in the works[35], the conclusion of which is as follows: in solutions of contact problems for elastic and rigid bodies, peculiarities of power order arise $O\left(\rho^{1-\gamma}\right)$, where ρ – distance from point to contact boundary, γ – the parameter expressed from some

[32] Guz A. N. Nonclassical Problems of Fracture/Failure Mechanics: On the Occasion of the 50th Anniversary of Research (Review). III. *I*nternational Applied Mechanics. 2019. Vol. 55. № 4. Pp. 343–415. https://doi.org/10.1007/s10778-019-00960-4

[33] Гузь А.Н., Рудницкий В.Б. Основы теории контактного взаимодействия упругих тел с начальными (остаточными) напряжениями. Хмельницький: вид. ПП Мельник, 2006. 710 с.

[34] Гузь А.Н., Рудницкий В.Б. Основы теории контактного взаимодействия упругих тел с начальными (остаточными) напряжениями. Хмельницький: вид. ПП Мельник, 2006. 710 с.

[35] Развитие теории контактных задач в СССР / под ред. Л. А. Галина. Москва : Наука, 1976. 494 с.

transcendent equation[36] depends on the elastic constant bodies in contact and the structure of the elastic potential. At such points, the stress from the contact interaction of bodies of physical content is not borne, but also the calculation of the integral characteristics of contact problems has no effect.

So, formulate the problem: Consider the elastic cylindrical stamp (Figure 1.) of radius *R* and height *H* with initial stresses, squeezed into the elastic layer under the action of force *P* after the initial deformation state occurs there.

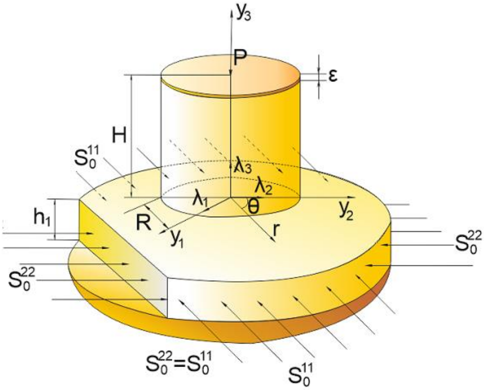

Fig. 1. Cylindrical stamp, layer and base with initial stresses

The thickness of the layer in the initial deformed state is related to the thickness in the undiformed state by the ratio $h_1 = \lambda_3 h_2$. We will count that the external load is applied only to the free end of the elastic die, under which all the points of the stamp move in the direction of the symmetry axis y_3 by the same value ε. We believe that the surfaces outside the contact area remain free from the influence of external forces, and there is no friction in the contact area, and the movements and stresses are continuous.

Suppose the initial state of the bodies is homogeneous, and the ratio[37]:

$$y_m = x_m + U_m^0,\ U_m^0 = \delta_{mi}(\lambda_m - 1)\lambda_i^{-1} y_i\ .\ (i,\ m = 1,2,3)$$

[36] Гузь А.Н., Рудницкий В.Б. Основы теории контактного взаимодействия упругих тел с начальными (остаточными) напряжениями. Хмельницький: вид. ПП Мельник, 2006. 710 с.

[37] Гузь А.Н., Рудницкий В.Б. Основы теории контактного взаимодействия упругих тел с начальными (остаточными) напряжениями. Хмельницький: вид. ПП Мельник, 2006. 710 с.

Then the basic equation in displacements for compressible bodies has the form:

$$L'_{m\alpha}U_\alpha = 0,\ \ L'_{m\alpha} = \omega'_{ij\alpha\beta}\,\partial^2\big/\partial y_i \partial y_\beta\ ,\ \ (i, m, \alpha, \beta = \overline{1, 3}) \tag{1}$$

and for incompressible bodies together with the incompressibility condition:

$$L'_{m\alpha}U_\alpha + q'_{\alpha m}\,\partial p'/\partial y_\alpha = 0,\ L'_{m\alpha} = \kappa'_{im\alpha\beta}\,\partial^2\big/\partial y_i \partial y_\beta\ ,$$
$$q'_{ij}\,\partial U_j\big/\partial y_i = 0,\ q'_{ij} = \lambda_i q_{ij},\ (i, j, m, \alpha, \beta = \overline{1, 3}). \tag{2}$$

Expressions for determining the components of the stress tensor for compressible and incompressible bodies are written as:

$$Q'_{ij} = \omega'_{ij\alpha\beta}\,\frac{\partial U_\alpha}{\partial y_\beta},\ Q'_{ij} = \kappa_{ij\alpha\beta}\,\frac{\partial U_\alpha}{\partial y_\beta} + q'_{ij}\,p,$$

$$\omega'_{ij\alpha\beta} = \frac{\lambda_i\lambda_\beta}{\lambda_1\lambda_2\lambda_3}\,\omega_{ij\alpha\beta},\ \kappa'_{ij\alpha\beta} = \frac{\lambda_i\lambda_\beta}{\lambda_1\lambda_2\lambda_3}\,\kappa_{ij\alpha\beta}\ ,$$

where $\omega'_{im\alpha\beta},\ \kappa'_{im\alpha\beta}$ – the components of the fourth order tensor of the elastic moduls,

$$p = (\lambda_1 q_1)^{-1}\left\{\left[\tilde\kappa_{1111} - \lambda_1 q_1(\lambda_3 q_3)^{-1}\left(\tilde\kappa_{1133} + \tilde\kappa_{1313}\right)\right]\Delta_1 + \tilde\kappa_{3113}\,\partial^2\big/\partial\,y_3^2\right\}\partial^2\big/\partial\,y_3^2\,\tilde\chi.$$

At homogeneous initial stresses $S_0^{11} = S_0^{22} \neq 0;\quad S_0^{33} = 0;\quad \lambda_1 = \lambda_2 \neq \lambda_3$ solutions of equations (1), (2) represent through cylindrical coordinates (r, θ, z_i) as solutions of the equation:

$$(\Delta_1 + \xi_2'^2\,\partial^2\big/\partial\,y_3^2)(\Delta_1 + \xi_3'^2\,\partial^2\big/\partial\,y_3^2)\tilde\chi = 0\,, \tag{3}$$

where $\Delta_1 = \partial^2\big/\partial r^2 + r^{-1}\,\partial/\partial r$.

Take into account the condition of existence the only solution of linearized theory of elasticity for compressed and uncompressed solids[38], there are two variants to represent the general solution (3): the case of equal radicals $(\xi_2'^2 = \xi_3'^2)$ and the case of different radicals $(\xi_2'^2 \neq \xi_3'^2)$.

In the case of equal radicals of equations (3):

$$\tilde\chi = \tilde\chi_1 + y_3\tilde\chi_2,\ \left(\Delta_1 + \xi_2'^2\,\partial^2\big/\partial\,y_3^2\right)\tilde\chi_1 = 0,\ \ \left(\Delta_1 + \xi_2'^2\,\partial^2\big/\partial\,y_3^2\right)\tilde\chi_2 = 0 \tag{4}$$

In the case of different radicals of equations (3):

$$\tilde\chi = \tilde\chi_1 + \tilde\chi_2,\ \left(\Delta_1 + \xi_2'^2\,\partial^2\big/\partial\,y_3^2\right)\tilde\chi_1 = 0,\ \ \left(\Delta_1 + \xi_3'^2\,\partial^2\big/\partial\,y_3^2\right)\tilde\chi_2 = 0 \tag{5}$$

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[38] Гузь А.Н., Рудницкий В.Б. Основы теории контактного взаимодействия упругих тел с начальными (остаточными) напряжениями. Хмельницький: вид. ПП Мельник, 2006. 710 с.

Harmonic functions $\tilde{\chi}_i$ $(i=1,2)$ present in the form of sum of the two others harmonic functions:

$$\tilde{\chi}_1 = \Phi_{11} + \Phi_{12}, \tag{6}$$

$$\tilde{\chi}_2 = \Phi_{21} + \Phi_{22}, \tag{7}$$

Which are the solutions of differential equation by Laplace:

$$\nabla^2\Phi_{kj} = 0, \quad \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + v_i^2 \frac{\partial^2}{\partial y_3^2}, \quad (k, j, i = 1, 2) \tag{8}$$

Partial solutions (2.38) – (2.39) of differential equations (2.40), take into account their limitations if $r = 0$, will find the dividing of variables (the method of Fure)[39] present as:

$$\Phi_{lj} = R_{lj} Z_{lj}, \quad (l, j = 1, 2) \tag{9}$$

The solution is Φ_{11}:

$$\Phi_{11} = R_{11}(r) Z_{11}(y_3), \tag{10}$$

Substitute (10) in (8) and divide by $v_i^2 R_{11} Z_{11}$, find

$$\frac{R_{11}'' + \frac{1}{r} R_{11}}{v_i^2 R_{11}} = -\frac{Z_{11}''}{Z_{11}} = \gamma_k^2 , \tag{11}$$

here γ_k^2 is the eigenvalue of the problem (constant).

Get the following equations

$$R_{11}'' + r^{-1} R_{11} - v_i^2 \gamma_k^2 R_{11} = 0, \quad Z_{11}'' + \gamma_k^2 Z_{11} = 0, \tag{12}$$

The first equation (12) is the Bessel equation[40], the solution of which will be written as

$$R_{11}(r) = A_k I_0(\gamma_k v_1 r) + \hat{B}_k K_0(\gamma_k v_1 r) \tag{13}$$

where $I_0(\gamma_k v_1 r),\ K_0(\gamma_k v_1 r) -$ McDonald's functions are of the second kind, $A_k, \hat{B}_k -$ uncertain coefficients.

The second equation (12) is an ordinary differential equation with constant coefficients where solution is

$$Z_{11}(\gamma_k v_1 z_1) = C_k \sin(\gamma_k v_1 z_1) + D_k \cos(\gamma_k v_1 z_1) \tag{14}$$

Such as the solution of the first equation (8) has to be limited if $r = 0$, so the constant $\hat{B}_k$ it is necessary to compare to 0, because $K_0(0) = \infty$.

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[39] Тихонов А. Н., Самарский А. А. Уравнения математической физики. Издание 3, испр. и доп. М.: Наука, 1966. 724 с.

[40] Ватсон Г. Н. Теория бесселевых функций. Часть 1. М., 1949. 798 с.

In order for the solution (14) to satisfy the specific boundary conditions of the problems which will be executed below, to solve the Sturm-Liouville problem[41] and find γ_k^2 . From here

$$\Phi_{11} = A_k I_0(\gamma_k v_i r) S_1(\gamma_k v_i z_i) \tag{15}$$

where $S_1(\gamma_k v_i z_i) = C_k \sin(\gamma_k v_i z_i) + D_k \cos(\gamma_k v_i z_i), \quad (i = 1, 2)$.

The function Φ_{12} is the same as (10), only (11) will look like

$$\frac{R_{12}'' + \frac{1}{r} R_{12}}{v_i^2 R_{12}} = -\frac{Z_{12}''}{Z_{12}} = -\alpha_k^2 \tag{16}$$

where α_k^2 – another value of the problem (constant value).

Omitting calculations (12) – (14), obsessed

$$\Phi_{12} = J_0(\alpha_k r) S_2(\alpha_k z_i) \tag{17}$$

where $S_2(\alpha_k z_i) = E_k sh(\alpha_k z_i) + F_k ch(\alpha_k z_i), \quad (i = 1, 2)$.

Similarly, we find solutions for the remaining functions (9).

Thus, the general solutions of the equation (3) for the cylindrical solid depending on the radicals (4 -5) of the defining equation (3) are:

For equal radicals $n_1 = n_2$ through y_3

$$\tilde{\chi}_1 = \sum_{k=1}^{\infty} \left[A_k I_0(\gamma_k v_1 r) S_1(\gamma_k y_3) + J_0(\alpha_k r) S_2\left(\frac{\alpha_k y_3}{v_1} \right) \right] \tag{18}$$

$$\tilde{\chi}_2 = \sum_{k=1}^{\infty} \left[B_k I_0(\gamma_k v_1 r) S_1(\gamma_k y_3) + J_0(\alpha_k r) S_3\left(\frac{\alpha_k y_3}{v_1} \right) \right]$$

Where the general solution (4) will be:

$$\tilde{\chi} = \sum_{k=1}^{\infty} \left[\left(A_k + \frac{y_3}{v_1} B_k \right) I_0(\gamma_k v_1 r) S_1(\gamma_k y_3) + J_0(\alpha_k r) \left(S_2\left(\frac{\alpha_k y_3}{v_1} \right) + \frac{y_3}{v_1} S_3\left(\frac{\alpha_k y_3}{v_1} \right) \right) \right] \tag{19}$$

And after replacing $z_i = \dfrac{y_3}{v_i}, \quad (i = \overline{1,2})$:

$$\tilde{\chi}_1 = \sum_{k=1}^{\infty} \left[A_k I_0(\gamma_k v_1 r) S_1(\gamma_k z_1 v_1) + J_0(\alpha_k r) S_2(\alpha_k z_1) \right] \tag{20}$$

$$\tilde{\chi}_2 = \sum_{k=1}^{\infty} \left[B_k I_0(\gamma_k v_1 r) S_1(\gamma_k z_1 v_1) + J_0(\alpha_k r) S_3(\alpha_k z_1) \right]$$

$$\tilde{\chi} = \sum_{k=1}^{\infty} \left[\left(A_k + z_1 B_k \right) I_0(\gamma_k v_1 r) S_1(\gamma_k z_1 v_1) + J_0(\alpha_k r) \left(S_2(\alpha_k z_1) + z_1 S_3(\alpha_k z_1) \right) \right] \tag{21}$$

For the different radicals $n_1 \neq n_2$ of the defining equation (3) through y_3

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[41] Бобик О. І., Бобик І. О., Литвин В.В. Рівняння математичної фізики: навчальний посібник. Львів:»Новий світ-2000», 2010. 256 с.

$$\tilde{\chi}_1 = \sum_{k=1}^{\infty}\left[A_k I_0(\gamma_k v_1 r)S_1(\gamma_k y_3) + J_0(\alpha_k r)S_2\left(\frac{\alpha_k y_3}{v_1}\right)\right]$$

$$\tilde{\chi}_2 = \sum_{k=1}^{\infty}\left[B_k I_0(\gamma_k v_2 r)S_1(\gamma_k y_3) + J_0(\alpha_k r)S_3\left(\frac{\alpha_k y_3}{v_2}\right)\right] \qquad (22)$$

The general solution (5) will be:

$$\tilde{\chi} = \sum_{k=1}^{\infty}\left\{\left[A_k I_0(\gamma_k v_1 r)S_1(\gamma_k y_3) + J_0(\alpha_k r)S_2(\frac{\alpha_k y_3}{v_1})\right] + \right.$$
$$\left. + \left[B_k I_0(\gamma_k v_2 r)S_1(\gamma_k y_3) + J_0(\alpha_k r)S_3(\frac{\alpha_k y_3}{v_2})\right]\right\} \qquad (23)$$

And after replacing $z_i = y_3 v_i^{-1}, \quad (i = \overline{1,2})$:

$$\tilde{\chi}_1 = \sum_{k=1}^{\infty}\left[A_k I_0(\gamma_k v_1 r)S_1(\gamma_k z_1 v_1) + J_0(\alpha_k r)S_2(\alpha_k z_1)\right] \qquad (24)$$

$$\tilde{\chi}_2 = \sum_{k=1}^{\infty}\left[A_k I_0(\gamma_k v_2 r)S_1(\gamma_k z_2 v_2) + J_0(\alpha_k r)S_3(\alpha_k z_2)\right]$$

$$\tilde{\chi} = \sum_{k=1}^{\infty}\left\{\left[A_k I_0(\gamma_k v_1 r)S_1(\gamma_k z_1 v_1) + B_k I_0(\gamma_k v_2 r)S_1(\gamma_k z_2 v_2)\right] + \right.$$
$$\left. + J_0(\alpha_k r)\left[S_2(\alpha_k z_1) + S_3(\alpha_k z_2)\right]\right\}. \qquad (25)$$

where $S_1 = C_k \sin(\gamma_k v_1 z_1) + D_k \cos(\gamma_k v_1 z_1), \quad S_2 = E_k sh(\alpha_k z_1) + F_k ch(\alpha_k z_1),$ $S_3 = N_k sh(\alpha_k z_1) + M_k ch(\alpha_k z_1),$ $A_k, B_k, C_k, D_k, E_k, F_k, N_k, M_k$ – some constant coefficients.

In a system of circular cylindrical coordinates (r, θ, z_i), where $z_i = v_i^{-1} y_3, \quad v_i = \sqrt{n_i}, \quad (i = \overline{1,\ 2}), \quad n_1 = \xi_2'^2,\ n_2 = \xi_3'^2,$ such statement corresponds to the boundary conditions:

1) At the end of the cylinder $z_i = Hv_i^{-1}$, де $v_i = \sqrt{n_i}$, $(i = \overline{1,\ 2})$:

$$u_3'^{(1)} = -\varepsilon, \quad Q_{3r}'^{(1)} = 0, \quad (0 \le r \le R) \qquad (26)$$

2) On the edge of the elastic layer in the contact area $z_i = 0$, $(i = \overline{1,\ 2})$:

$$u_3'^{(1)} = u_3'^{(2)}, \quad Q_{33}'^{(1)} = Q_{33}'^{(2)}, \quad Q_{3r}'^{(1)} = Q_{3r}'^{(2)} = 0, \quad (0 \le r \le R) \qquad (27)$$

3) On the edge of the elastic layer outside the contact area $z_i = 0$, $(i = \overline{1,\ 2})$:

$$Q_{33}'^{(2)} = 0, \quad Q_{3r}'^{(2)} = 0, \quad (R \le r < \infty) \qquad (28)$$

4) On the side surface of the elastic die $r=R$:

$$Q_{rr}'^{(1)} = 0, \quad Q_{3r}'^{(1)} = 0, \quad (0 \le z_i \le Hv_i^{-1}) \qquad (29)$$

5) On the lower surface of the layer, $z_i = -\lambda_3 h_2 v_i^{-1} = -Hv_i^{-1}$, $(i = \overline{1,\ 2})$:

$$u_3'^{(2)} = 0, \quad u_r'^{(2)} = 0, \quad (0 \le r < \infty); \tag{30}$$

Equilibrium condition

$$P = -2\pi R^2 \int_0^1 \rho Q_{33}^{(2)}(0,\rho)\, d\rho, \tag{31}$$

which establishes the connection between the settling of the end and the equivalent load P.

Adding elementary solutions (which also satisfy the harmonic equation (8)) to the results (18) – (21). The solution of the problem (26) – (31) will be searched as:

For equal radicals $n_1 = n_2$ **:**

$$\tilde{\chi}_1 = A_0 z_1 + B_0 + C_0 z_1 (3r^2 - 2z_1^2) + \sum_{k=1}^{\infty} \left[A_k I_0(\gamma_k v_1 r) S_1(\gamma_k z_1 v_1) + J_0(\alpha_k r) S_2(\alpha_k z_1) \right] \tag{32}$$

$$\tilde{\chi}_2 = A_0 z_1 + B_0 + C_0 z_1 (3r^2 - 2z_1^2) + \sum_{k=1}^{\infty} \left[B_k I_0(\gamma_k v_1 r) S_1(\gamma_k z_1 v_1) + J_0(\alpha_k r) S_3(\alpha_k z_1) \right]$$

General solution $\tilde{\chi} = \tilde{\chi}_1 + v_1 z_1 \tilde{\chi}_2$ looks like:

$$\tilde{\chi} = (1 + v_1 z_1)\left[A_0 z_1 + B_0 + C_0 z_1 \left(3r^2 - 2z_1^2\right)\right] + \tag{33}$$

$$+\sum_{k=1}^{\infty} \left[\left(A_k + v_1 z_1 B_k\right) I_0(\gamma_k v_1 r) S_1(\gamma_k z_1 v_1) + J_0(\alpha_k r)\left(S_2(\alpha_k z_1) + v_1 z_1 S_3(\alpha_k z_1)\right)\right]$$

For the different radicals $n_1 \ne n_2$

$$\tilde{\chi}_1 = A_0 (r^2 - 2z_1^2) + C_0 z_1 (3r^2 - 2z_1^2) + \sum_{k=1}^{\infty} \left[A_k I_0(\gamma_k v_1 r) S_1(\gamma_k z_1 v_1) + J_0(\alpha_k r) S_2(\alpha_k z_1) \right] \tag{34}$$

$$\tilde{\chi}_2 = A_0 (r^2 - 2z_2^2) + C_0 z_2 (3r^2 - 2z_2^2) + \sum_{k=1}^{\infty} \left[A_k I_0(\gamma_k v_2 r) S_1(\gamma_k z_2 v_2) + J_0(\alpha_k r) S_3(\alpha_k z_2) \right]$$

General solution $\tilde{\chi} = \tilde{\chi}_1 + \tilde{\chi}_2$ will be:

$$\tilde{\chi} = r^2 (2A_0 + 3C_0 (z_1 + z_2)) - 2A_0 (z_1^2 + z_2^2) - 2C_0 (z_1^3 + z_2^3) +$$

$$+\sum_{k=1}^{\infty} \left\{ \left[A_k I_0(\gamma_k v_1 r) S_1(\gamma_k z_1 v_1) + B_k I_0(\gamma_k v_2 r) S_1(\gamma_k z_2 v_2) \right] + J_0(\alpha_k r) \left[S_2(\alpha_k z_1) + S_3(\alpha_k z_2) \right] \right\}$$

where A_0, B_0, C_0 – some constant coefficients according to the boundary conditions (26) – (30).

3. Analytical method of solution

Satisfying the boundary conditions (26) – (30), to determine the stress and strain state in the elastic cylinder, where $n_1 \ne n_2$ will get the general solution of the defining equation (3):

$$\tilde{\chi} = 0{,}5\varepsilon \left\{ \theta_8^{-1}(r^2 - z_1^2 - z_2^2) - \chi_0 \left[r^2 \left(\theta_8^{-1} + (2H\theta_6)^{-1}(z_1 + z_2) \right) - \theta_8^{-1}(z_1^2 + z_2^2) - (2H\theta_6)^{-1}(z_1^3 + z_2^3) \right] \right\} - \tag{35}$$

$$-\sum_{k=1}^{\infty}\left\{b_3^{(k)}\left[s_0\frac{I_1(\gamma_k v_2 R)}{I_1(\gamma_k v_1 R)}I_0(\gamma_k v_1 r)\sin(\gamma_k z_1 v_1)+I_0(\gamma_k v_2 r)\sin(\gamma_k z_2 v_2)\right]-J_0(\alpha_k r)\left[\tilde S_2(\alpha_k z_1)+\tilde S_3(\alpha_k z_2)\right]\right\}\chi_k$$

And for the case of equal radicals $n_1 = n_2$, the general solution (4) of the defining equation (3) is:

$$\hat\chi=\varepsilon\left\langle v_1 z_1(1+z_1)\left[(m_2-1)^{-1}+\chi_0\left((1-m_2)^{-1}-2E(3H\theta_2)^{-1}(3r^2-2z_1^2)\right)\right]+\right. \quad (36)$$

$$+R\sum_{k=1}^{\infty}\chi_k\left[R(2\gamma_k)^{-1}b_1^{(k)}\left(H\left(1+\frac{s_0(1-I_0(v_1\gamma_k R))}{v_1\gamma_k R I_1(v_1\gamma_k R)}\right)+z_1\right)\right.$$

$$\left.\left.I_0(\gamma_k v_1 r)\sin(\gamma_k z_1 v_1)+J_0(\alpha_k r)\mu_k^{-1}\left(\tilde S_2(\alpha_k z_1)+z_1\tilde S_3(\alpha_k z_1)\right)\right]\right\rangle$$

where $J_v(x)$, $I_v(x)$ – are the Bessel functions of the real and imaginary argument,

$$s_0=\frac{1+m_2}{1+m_1},\ \theta_8=m_1n_1^{-1}+m_2n_2^{-1},\quad \theta_6=m_1v_1^{-3}+m_2v_2^{-3},$$

$$b_3^{(k)}=4\varepsilon R^2 J_0(\mu_k)\left[\frac{\tilde c_1-\tilde c_0}{\mu_k^2+(\gamma_k v_1 R)^2}-\frac{v_2}{v_1 s_0}\frac{\tilde c_2-\tilde c_0}{\mu_k^2+(\gamma_k v_2 R)^2}\right](v_1 H\gamma_k^3 I_1(\gamma_k v_2 R)\left[v_2 W_k(2)-v_1 s_0 W_k(1)\right])^{-1},$$

$$W(j)=\frac{(\tilde c_0-\tilde c_j)I_0(\gamma_k v_j R)}{I_1(\gamma_k v_j R)}+\frac{1-\tilde c_0}{\gamma_k v_j R},\ \theta_2=E\left(8m_1(1+H)n_1^{-1}-4Hv_1^{-1}+(1-m_2)R^2H^{-1}\right),$$

$$\tilde c_0=\begin{cases}\omega'_{1111}\omega'^{-1}_{1122};\\ \lambda_1 q_1(\lambda_3 q_3)^{-1}(\kappa'_{1133}+\kappa'_{1313})\kappa'^{-1}_{1122};\end{cases}\qquad \tilde c_i=\begin{cases}\lambda_3\omega'_{1133}m_i\omega'^{-1}_{1122}n_i^{-1};\\ (\kappa'_{1133}m_i-\kappa'_{3113})\kappa'^{-1}_{1122}n_i^{-1};\end{cases}\qquad (i=\overline{1,2})$$

$$b_1^{(k)}=J_0(\mu_k)\gamma_k t_{00}\left\langle t_{14}sh^2(\alpha_k Hv_1^{-1})ch(\alpha_k Hv_1^{-1})+\right.$$

$$+t_{11}\left[(1+m_1)sh^2(\alpha_k Hv_1^{-1})\left[t_{12}sh(\alpha_k Hv_1^{-1})+t_{13}\right]+\right.$$

$$+t_{11}\left[(1+m_1)sh^2(\alpha_k Hv_1^{-1})\left[t_{12}sh(\alpha_k Hv_1^{-1})+t_{13}\right]+ch(\alpha_k Hv_1^{-1})\times\right.$$

$$+ch(\alpha_k Hv_1^{-1})\begin{cases}t_{10}ch(\alpha_k Hv_1^{-1})\left[c_0 sh(\alpha_k Hv_1^{-1})+c_1\left(1-ch(\alpha_k Hv_1^{-1})\right)\right]+\\ +c_1(1+m_2)sh(\alpha_k Hv_1^{-1})(1-\end{cases}$$

$$\left.\left.-ch(\alpha_k Hv_1^{-1}))\right\}\right]\Big\rangle\Big/\Big\langle(I_0(\gamma_k v_1 R)-1)\begin{cases}t_{11}t_{21}t_{00}sh^2(\alpha_k Hv_1^{-1})+\\ +t_{22}\left[c_1t_{10}ch(\alpha_k Hv_1^{-1})\left(1+ch(\alpha_k Hv_1^{-1})\right)+\right.\end{cases}$$

$$\left.\left.+(1+m_1)sh(\alpha_k Hv_1^{-1})\left(t_{12}ch(\alpha_k Hv_1^{-1})+c_1s_0+t_{23}sh(\alpha_k Hv_1^{-1})\right)\right]\right\}+$$

$$+c_1\left(ch(\alpha_k Hv_1^{-1})-1\right)\left[t_{10}ch(\alpha_k Hv_1^{-1})+\right.$$

$$\left.+(1+m_2)sh(\alpha_k Hv_1^{-1})\right]+sh(\alpha_k Hv_1^{-1})\left[c_0t_{10}ch(\alpha_k Hv_1^{-1})-t_{24}sh(\alpha_k Hv_1^{-1})\right]\Big\rangle,$$

$$\tilde S_2(\alpha_k z_1)=Rs_0\mu_k^{-1}ch(\mu_k z_1R^{-1})+E^{(k)}sh(\mu_k z_1R^{-1}),$$

$$\tilde S_3(\alpha_k z_2)=-sh(\mu_k z_1R^{-1})-M^{(k)}ch(\mu_k z_1R^{-1}),\quad M^{(k)}=M_kN_k^{-1},\quad E^{(k)}=-E_kN_k^{-1}.$$

Then the expressions for the components of the displacement vector and the stress tensor for the cylindrical punch will be sought as:

For $n_1 \neq n_2$:

$$U_r'^{(1)} = \varepsilon\theta_+ r(2H\theta_6)^{-1}\chi_0 + \sum_{k=1}^{\infty}\Big\{\gamma_k^2 b_3^{(k)}\left[s_0 I_1(\gamma_k v_2 R)(I_1(\gamma_k v_1 R))^{-1} v_1 I_1(v_1\gamma_k r)\cos(\gamma_k z_1 v_1) - v_2 I_1(v_2\gamma_k r)\cos(\gamma_k z_2 v_2)\right] +$$
$$+\alpha_k^2 J_1(\alpha_k r)\left(\tilde{S}_4(\alpha_k z_1)v_1^{-1} + \tilde{S}_5(\alpha_k z_2)v_2^{-1}\right)\Big\}\chi_k$$

$$U_3'^{(1)} = -\varepsilon\left\{1+\chi_0\left[\frac{1}{H\theta_6}\left(\frac{m_1 z_1}{n_1}+\frac{m_2 z_2}{n_2}\right)-1\right]\right\} - \sum_{k=1}^{\infty}\Big\{\gamma_k^2 b_3^{(k)}\left[s_0\frac{I_1(\gamma_k v_2 R)}{I_1(\gamma_k v_1 R)} m_1 I_0(\gamma_k v_1 r)\sin(\gamma_k z_1 v_1) - m_2 I_0(\gamma_k v_2 r)\sin(\gamma_k z_2 v_2)\right] +$$
$$+\alpha_k^2 J_0(\alpha_k r)\left(\frac{m_1\tilde{S}_2(\alpha_k z_1)}{n_1} + \frac{m_2\tilde{S}_3(\alpha_k z_2)}{n_2}\right)\Big\}\chi_k$$

$$Q_{33}'^{(1)} = C_{44}(1+m_1)l_1\Big\langle -\frac{\varepsilon}{H\theta_6}\chi_0\left[\frac{1}{v_1}+\frac{s}{v_2}\right] - \tag{37}$$
$$-\sum_{k=1}^{\infty}\Big\{\gamma_k^3 b_3^{(k)}\left[s_0\frac{I_1(\gamma_k v_2 R)}{I_1(\gamma_k v_1 R)} n_1 I_0(\gamma_k v_1 r)\cos(\gamma_k z_1 v_1) - s n_2 I_0(\gamma_k v_2 r)\cos(\gamma_k z_2 v_2)\right] + \alpha_k^3 J_0(\alpha_k r)\left(\frac{\tilde{S}_4(\alpha_k z_1)}{v_1} + \frac{s\tilde{S}_5(\alpha_k z_2)}{v_2}\right)\Big\}\chi_k\Big\rangle$$

$$Q_{3r}'^{(1)} = C_{44}(1+m_1)\sum_{k=1}^{\infty}\Big\{s_0\gamma_k^3 b_3^{(k)}\left[v_2 I_1(\gamma_k v_2 r)\sin(\gamma_k z_2 v_2) - v_1 I_1(\gamma_k v_2 R)(I_1(\gamma_k v_1 R))^{-1} I_1(\gamma_k v_1 r)\sin(\gamma_k z_1 v_1)\right] +$$
$$+\alpha_k^3 J_1(\alpha_k r)\left[n_1^{-1}\tilde{S}_2(\alpha_k z_1) + s_0 n_2^{-1}\tilde{S}_3(\alpha_k z_2)\right]\Big\}\chi_k$$

For $n_1 = n_2$:

$$U_r'^{(1)} = \varepsilon\Big\langle\frac{4Ev_1}{H\theta_2}\chi_0 r\left(\frac{1}{v_1}+2z_1\right) - \sum_{k=1}^{\infty}\chi_k\Big\{\frac{R^2}{2}b_1^{(k)}\gamma_k v_1 I_1(v_1\gamma_k r)\Big[\Big(H\left(1+\frac{s_0(1-I_0(v_1\gamma_k R))}{v_1\gamma_k R I_1(v_1\gamma_k R)}\right) + v_1 z_1\Big)\cos(\gamma_k v_1 z_1) +$$
$$+\frac{\sin(\gamma_k v_1 z_1)}{\gamma_k} - J_1(\alpha_k r)\left[\frac{\alpha_k}{v_1}\left(\tilde{S}_4(\alpha_k z_1) + v_1 z_1\tilde{S}_5(\alpha_k z_1)\right) - \tilde{S}_3(\alpha_k z_1)\right]\Big]\Big\}\Big\rangle \tag{38}$$

$$U_3'^{(1)} = \varepsilon\Big\langle(m_2-1)v_1^{-1} + \left[1-2E(H\theta_2)^{-1}\left(r^2-2z_1^2+4m_1 z_1\left(v_1^{-1}+z_1\right)\right)\right]\chi_0 +$$
$$+\sum_{k=1}^{\infty}\chi_k\Big\{0,5R^2 b_1^{(k)}\gamma_k I_0(\gamma_k v_1 r)$$
$$\Big[\Big(H\left(1+s_0(1-I_0(v_1\gamma_k R))(v_1\gamma_k R I_1(v_1\gamma_k R))^{-1}\right) + v_1 z_1\Big)m_1\sin(\gamma_k v_1 z_1) +$$
$$+(1-m_2)\cos(\gamma_k v_1 z_1)\gamma_k^{-1}\Big] - J_0(\alpha_k r)n_1^{-1}$$
$$\left[m_1\alpha_k\left(\tilde{S}_2(\alpha_k z_1) + z_1 v_1\tilde{S}_3(\alpha_k z_1)\right) + (m_2-1)v_1\tilde{S}_5(\alpha_k z_1)\right]\Big\}\Big\rangle$$

$$Q_{33}'^{(1)} = C_{44}\varepsilon\Big\langle -8Ev_1(H\theta_2 R^2)^{-1}\chi_0\left[(1+m_1)l_1\left(v_1^{-1}+z_1\right) + (1+m_2)l_2 z_1\right] +$$
$$+\sum_{k=1}^{\infty}\chi_k\Big\{0,5R^2 b_1^{(k)}\gamma_k n_1 I_0(\gamma_k v_1 r)\Big[(1+m_1)l_1\gamma_k$$
$$\Big(H\left(1+s_0(1-I_0(v_1\gamma_k R))(v_1\gamma_k R I_1(v_1\gamma_k R))^{-1}\right) + v_1 z_1\Big)\cos(\gamma_k v_1 z_1) +$$
$$+(1+m_2)l_2\sin(\gamma_k v_1 z_1)\Big] - \alpha_k J_0(\alpha_k r)$$
$$\left[(1+m_1)l_1\alpha_k v_1^{-1}\left(\tilde{S}_4(\alpha_k z_1) + v_1 z_1\tilde{S}_5(\alpha_k z_1)\right) + (1+m_2)l_2\tilde{S}_3(\alpha_k z_1)\right]\Big\}\Big\rangle$$

$$Q'^{(1)}_{3r} = C_{44}\varepsilon \left\langle \frac{4Er\left(1+m_2\right)}{H\theta_2}\chi_0 + \sum_{k=1}^{\infty}\chi_k \left\{ \frac{R^2}{2}\gamma_k v_1 b_1^{(k)} I_1(\gamma_k v_1 r) \left[\frac{\left(1+m_1\right)\gamma_k}{\left(H\left(1 + \frac{s_0(1 - I_0(v_1\gamma_k R))}{v_1\gamma_k R I_1(v_1\gamma_k R)} \right) + v_1 z_1 \right)} \sin(\gamma_k v_1 z_1) - \right.\right.\right.$$

$$\left.\left.\left. - \left(1+m_2\right)\cos(\gamma_k v_1 z_1) \right] + \frac{\alpha_k}{v_1} J_1(\alpha_k r) \left[\alpha_k \left(1+m_1\right)\left(\tilde{S}_2\left(\alpha_k z_1\right) + v_1 z_1 \tilde{S}_3\left(\alpha_k z_1\right)\right) + \left(1+m_2\right)\tilde{S}_5\left(\alpha_k z_1\right) \right] \right\} \right\rangle$$

where $\theta_+ = v_1^{-1} + v_2^{-1}$,

$$\tilde{S}_2(\alpha_k z_1) = R^2\varepsilon\mu_k^{-2}\left[ch(\alpha_k z_1) - cth(\mu_k l v_1^{-1}) sh(\alpha_k z_1)\right], \tilde{S}_4(\alpha_k z_1) = R^2\varepsilon\mu_k^{-2}\left[sh(\alpha_k z_1) - cth(\mu_k l v_1^{-1}) ch(\alpha_k z_1)\right],$$

$$\tilde{S}_3(\alpha_k z_2) = \frac{n_2 R^2 \varepsilon}{n_1 \mu_k^2 s_0}\left[cth(\mu_k l v_2^{-1}) sh(\alpha_k z_2) - ch(\alpha_k z_2)\right],\ \tilde{S}_5(\alpha_k z_2) = \frac{n_2 R^2 \varepsilon}{n_1 \mu_k^2 s_0}\left[cth(\mu_k l v_2^{-1}) ch(\alpha_k z_2) - sh(\alpha_k z_2)\right].$$

The stress and strain state in the elastic layer with initial (residual) stresses is determined from[42] through the harmonic functions in the form of Hankel integrals. Satisfying the third condition (27), the second – (28) and the condition (30), after the number of transformations will have

for the different radicals $n_1 \neq n_2$

$$u'^{(2)}_r = -\hat{T}^3(\Omega_+^2; S_1^1; K_0^1; s_3; 1; 1),\quad u'^{(2)}_3 = -m_1 v_1^{-1}\hat{T}^3(\Omega_-^2; S_1^0; K_0^0; s_3; s_2; 1) \qquad (39)$$

$$Q'^{(2)}_{33} = C_{44}(1+m_1) l_1 R^{-1}\hat{T}^3(\Omega_+^2; S_2^0; K_1^0; s_3; s; 1),$$

$$Q'^{(2)}_{3r} = C_{44}(1+m_1) s_3 (v_1 R)^{-1}\hat{T}^3(\Omega_-^2; S_2^1; K_1^1; 1; 1; 1)$$

For the equal radicals $n_1 = n_2$:

$$u'^{(2)}_r = \varepsilon(\pi\theta_3)^{-1}\tilde{T}^1(\Omega_+^3; S_1^1; N_0^1; K_0^1; 1),$$

$$u'^{(2)}_3 = m_1\varepsilon(\pi\theta_3 v_1)^{-1}\tilde{T}^1(\Omega_-^3; S_1^0; N_0^0; K_0^0; s_1), \qquad (40)$$

$$Q'^{(2)}_{33} = (1+m_1)\varepsilon l_1 C_{44}(\pi\theta_3 R)^{-1}\tilde{T}^1(\Omega_+^3; S_2^0; N_1^0; K_1^0; s),$$

$$Q'^{(2)}_{3r} = -(1+m_1)\varepsilon C_{44}(\pi\theta_3 R v_1)^{-1}\tilde{T}^1(\Omega_-^3; S_2^1; N_1^1; K_1^1; s_0),$$

where $s_1 = \frac{m_1 - 1}{m_1},\quad s_2 = \frac{m_2}{m_1}\frac{v_1}{v_2},\quad s_3 = s_0\frac{v_1}{v_2},\quad s = s_0\frac{l_2}{l_1},$

$$\hat{T}^3(\Omega_\pm^{n_1}; S_{m_1}^n; K_{m_2}^n; \bar{\beta}_1; \bar{\beta}_2; l_1) = \frac{1}{\pi}\sum_{j=0}^{\infty} C_j^* \left[l_1 \Omega_\pm^{n_1}(S_{j+m_1}^n; 1; 0; \bar{\beta}_1; \bar{\beta}_2; 0) + \frac{1}{h}\sum_{i=1}^{\infty} a_i \Omega_\pm^{n_1}(S_{j+m_1}^n; h; 0; \bar{\beta}_1; \bar{\beta}_2; k_i) \right] +$$

$$\frac{\varepsilon}{\pi\theta_3}(\chi_0 - 1)\left[l_1 \Omega_\pm^{n_1}(S_{m_1}^n; 1; 0; \bar{\beta}_1; \bar{\beta}_2; 0) + \sum_{i=1}^{\infty} a_i \Omega_\pm^{n_1}(S_{m_1}^n; h; 0; \bar{\beta}_1; \bar{\beta}_2; k_i) \right] -$$

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[42] Ярецька Н.О. Розв’язок контактної задачі для попередньо напружених циліндричного штампа та шару, що лежить без тертя на основі без початкових напружень. *Вісник Запорізького національного університету. Фізико-математичні науки*. Запоріжжя: Видавничий дім «Гельветика». 2021. № 1. С. 90-100. https://doi.org/10.26661/2413-6549-2021-1-11

$$-\frac{\varepsilon\theta_4}{\pi\theta_3}\sum_{j=1}^{\infty}\chi_j\left[l_1\Omega_{\pm}^{n_1}(K_{m_2}^{n};1;\mu_j;\bar{\beta}_1;\bar{\beta}_2;0)+\frac{1}{h}\sum_{i=1}^{\infty}a_i\Omega_{\pm}^{n_1}(K_{m_2}^{n};h;\mu_j;\bar{\beta}_1;\bar{\beta}_2;k_i)\right],$$

$$\Omega_{\pm}^{2}(\hat{\mathrm{L}}_m^n,\mathrm{t},\mu,\mathrm{k},\mathrm{a},\theta)=k\left[\hat{\mathrm{L}}_m^n\left(\frac{\rho}{t},\mu,-\frac{z_1}{tR}+\theta\right)\pm\hat{\mathrm{L}}_m^n\left(\frac{\rho}{t},\mu,\frac{z_1}{tR}+\frac{h_1}{tRv_1}+\theta\right)+\hat{\mathrm{L}}_m^n\left(\frac{\rho}{t},\mu,-\frac{z_1}{tR}-\frac{h_1}{tRv_1}+\theta\right)\right]-$$

$$-\mathrm{a}\left[\hat{\mathrm{L}}_m^n\left(\frac{\rho}{t},\mu,\frac{z_2}{tR}+\theta\right)\pm\hat{\mathrm{L}}_m^n\left(\frac{\rho}{t},\mu,\frac{z_2}{tR}+\frac{h_1}{tRv_2}+\theta\right)+\hat{\mathrm{L}}_m^n\left(\frac{\rho}{t},\mu,-\frac{z_2}{tR}-\frac{h_1}{tRv_2}+\theta\right)\right]$$

$$,\quad \hat{\mathrm{L}}_m^n(\mathrm{t},0,\mathrm{u})=\hat{\mathrm{L}}_m^n(\mathrm{t},\mathrm{u})\,.$$

$$\tilde{T}^1(\Omega_{\pm}^{l};S_{m_1}^{n};N_{m_2}^{n};K_{m_3}^{n};\mathrm{k};\mathrm{a})=(1+a_0)\Big\langle(1-\chi_0)\cdot\Omega_{\pm}^{l}\left(S_{m_1}^{n};0;\mathrm{k};\mathrm{a};0\right)-$$

$$-\frac{\theta_3}{\varepsilon}\sum_{j=0}^{\infty}C_j^{**}\Omega_{\pm}^{l}\left(S_{j+m_1}^{n};0;\mathrm{k};\mathrm{a};0\right)-$$

$$-\sum_{k=1}^{\infty}\chi_k\Omega_{\pm}^{l}\left(K_{m_3}^{n};\mu_k;\mathrm{k};\mathrm{a};0\right)-\frac{2(m_2-1)R^2}{\theta_2}\chi_0\Omega_{\pm}^{l}\left(N_{m_2}^{n};0;\mathrm{k};\mathrm{a};0\right)+$$

$$+\theta_4\frac{(m_2-1)R^2}{2}\sum_{k=1}^{\infty}b_1^{(k)}\chi_k\Omega_{\pm}^{l}\left(K_{m_3}^{n};i\gamma_k v_1R;\mathrm{k};\mathrm{a};0\right)\Big\rangle+\sum_{\tau=1}^{\infty}a_{\tau}\Big\langle(1-\chi_0)\Omega_{\pm}^{l}\left(S_{m_1}^{n};0;\mathrm{k};\mathrm{a};v_1\tau\right)-$$

$$-\frac{\theta_3}{\varepsilon}\sum_{j=0}^{\infty}C_j^{**}\Omega_{\pm}^{l}\left(S_{j+m_1}^{n};0;\mathrm{k};\mathrm{a};v_1\tau\right)-\frac{2(m_2-1)R^2}{\theta_2}\chi_0\Omega_{\pm}^{l}\left(N_{m_2}^{n};0;\mathrm{k};\mathrm{a};v_1\tau\right)+$$

$$+\theta_4\sum_{k=1}^{\infty}\chi_k\Omega_{\pm}^{l}\left(K_{m_3}^{n};\mu_k;\mathrm{k};\mathrm{a};v_1\tau\right)+\frac{(m_2-1)R^2}{2}\sum_{k=1}^{\infty}b_1^{(k)}\chi_k\Omega_{\pm}^{l}\left(K_{m_3}^{n};i\gamma_k v_1R;\mathrm{k};\mathrm{a};v_1\tau\right)\Big\rangle$$

$$\Omega_{\pm}^{3}(\hat{\mathrm{L}}_m^n,\mu,\mathrm{k},\theta)=(A_1^{02}-s_0)\,\hat{\mathrm{L}}_m^n\left(\rho,\mu,z_1R^{-1}-\theta\right)+$$

$$+\left(A_1^{12}-2k\right)\hat{\mathrm{L}}_m^n\left(\rho,\mu,2h_1(Rv_1)^{-1}+z_1R^{-1}-\theta\right)+$$

$$+\left(A_1^{22}-2k\right)\hat{\mathrm{L}}_{m+1}^n\left(\rho,\mu,2h_1(Rv_1)^{-1}+z_1R^{-1}-\theta\right)\pm$$

$$\pm\left(A_1^{02}+s_0-2k\right)\hat{\mathrm{L}}_m^n\left(\rho,\mu,-z_1R^{-1}-\theta\right)\pm2z_1R^{-1}\,\hat{\mathrm{L}}_{m+1}^n\left(\rho,\mu,-z_1R^{-1}-\theta\right)\pm$$

$$\pm\left(A_1^{12}-2k\right)\hat{\mathrm{L}}_m^n\left(\rho,\mu,2h_1(Rv_1)^{-1}-z_1R^{-1}-\theta\right)\pm$$

$$\pm(A_1^{22}+2z_1R^{-1})\,\hat{\mathrm{L}}_{m+1}^n\left(\rho,\mu,2h_1(Rv_1)^{-1}-z_1R^{-1}-\theta\right),$$

$$\hat{\mathrm{L}}_m^n(\mathrm{t},0,\mathrm{u})=\hat{\mathrm{L}}_m^n(\mathrm{t},\mathrm{u}),\ A_1^{02}=s_0(s_0-s_1)(s_0+s_1)^{-1},\ A_1^{12}=2s_0(s_0-s_1)(s_0+2s_1)(s_0+s_1)^{-2},$$

$$\mathrm{A}_1^{22}=h_1(s_0v_1R)^{-1}\mathrm{A}_1^{12},\ S_n^m\left(\rho;z\right)=\int_0^{\infty}\eta^{n-2}\sin\eta e^{-z\cdot\eta}J_m(\eta\rho)d\eta,$$

$$K_n^m\left(\rho;\mu_k;z\right)=\int_0^{\infty}\eta^n\psi_0(\eta,\mu_k)e^{z\cdot\eta}J_m(\eta\rho)d\eta\,,$$

$N_n^m(\rho;z)=\int_0^\infty \eta^n\psi_1(\eta,0)e^{z\cdot\eta}J_m(\eta\rho)d\eta$, C_j^{**} – coefficients of the division in a line $F(\eta)$, k_i, a_i – some constants (i=0,1,2,...).

A value of coefficients $n_i,\ \ m_i,\ \ c_{44},\ \ l_i$, D_{44} in the case of equal radicals:

$$m_1=\begin{cases}(\omega'_{1111}n_1-\omega'_{3113})(\omega'_{1133}+\omega'_{1313})^{-1};\\ \lambda_1 q_1 n_1(\lambda_3 q_3)^{-1};\end{cases}\quad D_{44}=\begin{cases}\omega'_{1122},\\ \kappa'_{1122}.\end{cases}$$

$$l_1=\begin{cases}\omega'^{-1}_{1313}(\omega'_{1331}+(\omega'_{1313}-\omega'_{1331})(\omega'_{1133}+\omega'_{1313})(\omega'_{1111}n_1+\omega'_{1133})^{-1});\\ \kappa'^{-1}_{1313}(\kappa'_{1331}+\lambda_3 q_3(\kappa'_{1313}-\kappa'_{1331})(\lambda_3 q_3+\lambda_1 q_1 n_1)^{-1});\end{cases}$$

$$m_2=\begin{cases}(\omega'_{1133}-\omega'_{1313})(\omega'_{1133}+\omega'_{1313})^{-1},\\ 1,\end{cases}$$

$$l_2=\begin{cases}(\omega'_{3333}(m_1+m_2-1)-\omega'_{1133}n_1)(n_1\omega'_{1313}(1+m_2))^{-1},\\ (\kappa'_{3333}m_1+(\lambda_1^{-1}q_1^{-1}\lambda_3 q_3\kappa'_{1111}-2\kappa'_{1133}-\kappa'_{1313})n_1-3\lambda_1^{-1}q_1^{-1}\lambda_3 q_3\kappa'_{3113})(2n_1\kappa'_{1313})^{-1},\end{cases}$$

$$m_1=\begin{cases}\dfrac{\omega'_{1111}n_1-\omega'_{3113}}{\omega'_{1133}+\omega'_{1313}};\\[2mm] \dfrac{\lambda_1 q_1}{\lambda_3 q_3}n_1;\end{cases}\quad l_1=\begin{cases}\dfrac{\omega'_{1331}}{\omega'_{1313}}+\dfrac{\omega'_{1313}-\omega'_{1331}}{\omega'_{1313}}\dfrac{\omega'_{1133}+\omega'_{1313}}{\omega'_{1111}n_1+\omega'_{1133}};\\[2mm] \dfrac{\kappa'_{1331}}{\kappa'_{1313}}+\dfrac{\kappa'_{1313}-\kappa'_{1331}}{\kappa'_{1313}}\dfrac{\lambda_3 q_3}{\lambda_3 q_3+\lambda_1 q_1 n_1};\end{cases}\quad C_{44}=\begin{cases}\omega'_{1313},\\ \kappa'_{1313}.\end{cases}$$

,

And in the case of different radicals:

$$D_{44}=\begin{cases}\omega'_{1122},\\ \kappa'_{1122}.\end{cases}\ C_{44}=\begin{cases}w'_{1313},\\ \kappa'_{1313}.\end{cases}\ m=(m_1+m_2-1)m_1^{-1},$$

$$m_i=\begin{cases}\dfrac{\omega'_{1111}n_i-\omega'_{3113}}{\omega'_{1133}+\omega'_{1313}};\\[2mm] \dfrac{\lambda_1 q_1}{\lambda_3 q_3}n_i;\end{cases}\quad l_i=\begin{cases}\dfrac{\omega'_{1331}}{\omega'_{1313}}+\dfrac{\omega'_{1313}-\omega'_{1331}}{\omega'_{1313}}\dfrac{\omega'_{1133}+\omega'_{1313}}{\omega'_{1111}n_i+\omega'_{1133}};\\[2mm] \dfrac{\kappa'_{1331}}{\kappa'_{1313}}+\dfrac{\kappa'_{1313}-\kappa'_{1331}}{\kappa'_{1313}}\dfrac{\lambda_3 q_3}{\lambda_3 q_3+\lambda_1 q_1 n_i};\end{cases}$$

for compressive solids:

$$n_{1,2}=c'\pm\sqrt{c'^2-\frac{\omega'_{3113}\omega'_{3333}}{\omega'_{1331}\omega'_{1111}}},\qquad 2c'\omega'_{1111}\omega'_{1331}=\omega'_{1111}\omega'_{3333}+\omega'_{1331}\omega'_{3113}-\left(\omega'_{1133}+\omega'_{1313}\right)^2,$$

for uncompressible solids:

$$n_{1,2}=c'\pm\sqrt{c'^2-\frac{\lambda_3\cdot q_3^2\kappa'_{3113}}{\lambda_1^2 q_1^2\kappa'_{1331}}},\quad 2c'\kappa'_{1331}=\kappa'_{3333}+\frac{\lambda_3^2 q_3^2}{\lambda_1^2 q_1^2}\kappa'_{1111}-\frac{2\lambda_3\cdot q_3}{\lambda_1 q_1}\left(\kappa'_{1133}+\kappa'_{1313}\right).$$

Using the solutions for cylinder (35) – (38) and satisfying third condition (27) and second condition (29), find the eigenvalues of the problem (26) – (31)

for the case of uneven roots $n_1 \neq n_2$:

$$\gamma_k = \frac{\pi(2k+1)}{H}, \quad \alpha_k = \frac{\mu_k}{R}, \text{ де } J_1(\mu_k) = 0. \qquad (41)$$

For the case of equal roots $n_1 = n_2$**:**

$$\alpha_k = \frac{\mu_k}{R}, \ \gamma_k = 2\pi k H^{-1}, \quad (k = 0, 1, 2, ...) \qquad (42)$$

With the help of the first conditions (27) and (28) it is possible to define the unknown function *F(η)* of equal integral equations for different radicals

$$\int_0^\infty F(\eta)\eta^{-1} J_0(\eta\rho) d\eta = f(\rho), \ (\rho < 1), \quad \int_0^\infty F(\eta) J_0(\eta\rho) d\eta = 0, \ (\rho > 1), \qquad (43)$$

for the case of different radicals $n_1 \neq n_2$

$$f(\rho) = \frac{\varepsilon}{\theta_3}\left(\chi_0 - 1 - \theta_4 \sum_{k=1}^{\infty} \chi_k J_0(\mu_k \rho) + \frac{\theta_3}{\varepsilon}\int_0^\infty \frac{F(\eta)}{\eta} G(\eta h) J_1(\eta\rho) d\eta\right),$$

$$\theta_4 = \frac{v_1(m_2 - 1) - m_1 s_0}{n_1}, \ \theta_3 = \frac{m_1}{v_1}(s_1 - s_0),$$

For the equal radicals $n_1 = n_2$**:**

$$f(\rho) = -\varepsilon\theta_3^{-1}\left(1 - \chi_0 - 2(m_2 - 1)\frac{R^2}{\theta_2}\chi_0\rho + \right.$$

$$\left. +\theta_4 \sum_{k=1}^{\infty} \chi_k J_0(\mu_k \rho) + +0,5(m_2 - 1) R^2 \sum_{k=1}^{\infty} b_1^{(k)} \chi_k I_0(\gamma_k v_1 R\rho)\right)$$

$$+\int_0^\infty \eta^{-1} F(\eta) G(\eta h) J_1(\eta\rho) d\eta$$

where $\theta_3 = m_1(s_1 - s_0)v_1^{-1}$.

Moreover, for the case of equal radicals, the type of function $G(\eta)$ and quantity of κ are determined from boundary conditions (26) – (31) and tasks:

Task 1. For the layer with initial stresses lying frictionless on the non-deformable base**:**

$$G(t) = (t - e^{-t} + 1)(sht)^{-1} P(t), \quad P(t) = \kappa \cdot sht(t + \kappa \cdot sht)^{-1}, \quad \kappa = s_0 - s, \quad t = 2h\eta v_1^{-1}. \qquad (44)$$

So, for the (44), it is possible to approximate the function by expression

So, for the (44) and $n_1 = n_2$, it is possible to approximate the function $P(x)$ by expression

$$P(x) \cong 1 - (\tilde{\vartheta}_1 + 1)^{-1} x (shx)^{-1} - 0,14x(\tilde{\vartheta}_1(\tilde{\vartheta}_1 + 1)chx)^{-1},$$

where $\tilde{\vartheta}_1 = s - s_0$.

Task 2. For the layer with initial stresses that is rigidly connected to the non-deformable base:

$$G(t)=\left(1-t+0,5\kappa t^2cht+(1-s_1)sht\right)P(t),\quad \kappa=(s_0-s_1)(1-s)-(1-s_0)(s_1-s),\qquad (45)$$

$$P(t)=1+0,5\kappa t^2+\kappa(1-s_1)(s_0-s)cht,\quad t=h\eta v_1^{-1}.$$

Task 3. For the layer with initial stresses lying frictionless on the elastic base with initial stress:

$$G(t)=\left[\begin{array}{l}t+(\beta k(s_1-s_0)-\alpha k_1(s-s_0)+\\+\beta kt)sh^2t+0,5\left((\beta k(s_1-s_0)(s-s_0)^{-1}-\alpha k_1)(t+(s-s_0))+\right.\end{array}\right.$$

$$\left.\left.+(\alpha k_1-\beta k)(s-s_0)^{-1}t^2\right)sht-\left(\begin{array}{l}\beta k(s_1-s_0)(s-s_0)^{-1}+\\+(\alpha k_1-\beta k)(s-s_0)^{-1}t\end{array}\right)tch^2t\right]P(t),\qquad (46)$$

$$\begin{aligned}P(t)=&(2(s-s_0)\alpha k_1+\beta k(s_1-s_0)t)sh^2t-0,5(\beta k(s_1-s_0)+\\&+(\beta k(s_1-s_0)(s-s_0)^{-1}-\alpha k_1)t+\\+&\begin{array}{l}(\alpha k_1-\beta k)(s-s_0)^{-1}t^2)sh2t-(\beta k(s_1-s_0)(s-s_0)^{-1}+\\+(\alpha k_1-\beta k)(s-s_0)^{-1}t)tcht,\end{array}\qquad \kappa=1,\quad t=h\eta v_1^{-1}.\end{aligned}$$

And for the different radicals:

Task 1. For the layer with initial stresses lying frictionless on the non-deformable base:

$$\begin{aligned}G(\eta h)=&(sh(\eta h\theta_+)+(s_2-s_3)\kappa^{-1}sh(\eta h\theta_-)+ch(\eta h\theta_-)-\\&-sh(\eta h\theta_+))(sh(\eta h\theta_+))^{-1}P(\eta h),\end{aligned}\qquad (47)$$

$$P(\eta h)=1+(s+s_3)\kappa^{-1}(sh(\eta h\theta_+))^{-1}sh(\eta h\theta_--1),\quad \kappa=1+s_0.$$

Moreover, for (47), in $n_1\neq n_2$ can be used even an approximate expression for $P(x)$

$$\begin{aligned}P(x)\cong&1-\tilde\kappa(\tilde\kappa\,\theta^-/\theta^++1)^{-1}sh(x\theta^-)(sh(x\theta^+))^{-1}-\\&-0,28\tilde\kappa^2(1+\tilde\kappa^2\,\theta^-/\theta^+)^{-1}sh(x\theta^-)(sh(x\theta^+))^{-1},\end{aligned}$$

where $\theta^\pm=v_1^{-1}\pm v_2^{-1}$.

Task 2. For the layer with initial stresses that is rigidly connected to the non-deformable base:

$$G(\eta h)=\left(\begin{array}{l}1+\dfrac{(s_3-s)(s_2-1)}{\kappa}ch(\eta h\theta_+)-\dfrac{(s_3+s)(s_2+1)}{\kappa}ch(\eta h\theta_-)+\\+(s_2-1)sh(\eta h\theta_+)-(s_2+1)sh(\eta h\theta_-)\end{array}\right)P(\eta h),$$

$$P(\eta h)=\left(\begin{array}{l}1+(s_3-s)(s_2-1)\kappa^{-1}ch(\eta h\theta_+)-\\-(s_3-s)(s_2+1)\kappa^{-1}ch(\eta h\theta_-)\end{array}\right)^{-1},\quad \kappa=2(s_2+s\cdot s_3).\qquad (48)$$

Task 3. For the layer with initial stresses lying without friction on the elastic base with initial stresses:

$$\begin{aligned}G(\eta h)=&(((s_2-s_3)(s-s_3)\beta k-s\alpha k_1)sh(\eta h\theta_+)+\\&+((s_2-s_3)(s+s_3)\beta k+s\alpha k_1)sh(\eta h\theta_-)+\end{aligned}$$

$$+(((s-s_3)^2+s_3)\alpha k_1-\beta k(s_2-s_3))ch(\eta h\theta_+)-$$
$$-(((s^2-s_3^2)-s_3)\alpha k_1-\beta k(s_2-s_3))ch(\eta h\theta_-)+$$
$$+2s\cdot s_3\alpha k_1)P(\eta h),\quad \kappa=1, \qquad (49)$$
$$P(\eta h)=2s\cdot s_3\alpha k_1+\beta k((s_2-s_3)(s-s_3)sh(\eta h\theta_+)+$$
$$+(s_2-s_3)(s+s_3)sh(\eta h\theta_-))-$$
$$-\alpha k_1((s-s_1)^2 ch(\eta h\theta_+)+(s^2-s_3^2)ch(\eta h\theta_-)).$$

In cases of tasks 2, 3 functions (45), (46), (48), (49) are quite huge, which greatly complicates the calculation, so it will be difficult to calculate the displacements and stresses in the future numerically.

Applying the reference formula to (39) leads to Fredholm's integral equation of the second kind with relatively to the function *F(η)*

In the case of the different radicals $n_1 \neq n_2$:

$$\frac{F(\eta)}{\eta}=\frac{2\varepsilon}{\pi\theta_3}\left((\chi_0-1)\psi_0(\eta,0)-\theta_4\sum_{k=1}^{\infty}\chi_k\psi_0(\eta,\mu_k)+\frac{\theta_3}{\varepsilon}\int_0^{\infty}\frac{F(u)}{u}G(uh)\psi_0(\eta,u)du\right), \qquad (50)$$

For the equal radicals $n_1 = n_2$:

$$\frac{F(\eta)}{\eta}=-\frac{2\varepsilon}{\pi\theta_3}\left((1-\chi_0)\psi_0(\eta,0)-2(m_2-1)\frac{R^2}{\theta_2}\chi_0\psi_1(\eta,0)+\theta_4\sum_{k=1}^{\infty}\chi_k\psi_0(\eta,\mu_k)+\right.$$
$$\left.+0,5(m_2-1)R^2\sum_{k=1}^{\infty}b_1^{(k)}\chi_k\psi_0(\eta,i\gamma_k v_1 R)\right)+2\pi^{-1}\int_0^{\infty}u^{-1}F(u)G(uh)\psi_0(\eta,u)du \qquad (51)$$

where $\psi_n(x,y)=\int_0^1 t^n\cos xt\cos yt dt$.

Satisfying the second boundary condition (27), the solution (50) – (51) will researched by the method of consecutive approximations, taking the function for a zero approximation

$$F^{(0)}(\eta)/\eta=2\varepsilon(\pi\theta_3)^{-1}p(\eta),$$

where for the case of the different radicals $n_1 \neq n_2$:

$$p(\eta)=(\chi_0-1)\psi_0(\eta,0)-\theta_4\sum_{k=1}^{\infty}\chi_k\psi_0(\mu_k\eta).$$

for the case of equal radicals $n_1 = n_2$:

$$p(\eta)=\varepsilon\left((1-\chi_0)\psi_0(\eta,0)-2(m_2-1)R^2\theta_2^{-1}\chi_0\psi_1(\eta,0)+\theta_4\sum_{k=1}^{\infty}\chi_k\psi_0(\eta,\mu_k)+\right.$$
$$\left.+0,5(m_2-1)R^2\sum_{k=1}^{\infty}b_1^{(k)}\chi_k\psi_0(\eta,i\gamma_k v_1 R)\right)$$

The following approximations are determined by the formula

$$\frac{F^{(j)}(\eta)}{\eta}=\frac{2}{\pi}\int_0^\infty \frac{F^{(j-1)}(u)}{u}G(uh)J_0(\eta u)du$$

The solution will be written (37) as

$$F(\eta)=\sum_{n=0}^{\infty}F^{(n)}(\eta)\,. \qquad 52)$$

Note, that the process of consecutive approximations (52) converges if $h>1$ and $\lambda_1>\lambda_{kp}$, taking into account the studies conducted[43].

Satisfying the first of two boundary conditions (27) taking into account the orthogonality of the Besselian functions $J_0(\mu_k\rho)$, to determine the constants $\chi_i \quad (i=0,1,2,..)$ will be gotten the infinite quasiregular system of algebraic equations

$$\vartheta_k\chi_k+\sum_{n=0}^{\infty}\vartheta_{kn}\chi_n=\varpi_k \quad (k=0,1,2,..)\,. \qquad (53)$$

The coefficients of the system can be represented as:

where for the case of the different radicals $n_1\neq n_2$:

$$\vartheta_0=\varpi_0=\frac{2}{\pi}\left[1+\frac{2}{\pi}\sum_{j=1}^{\infty}\int_0^\infty\frac{\sin u}{u}G(hu)\psi_{j-1}(u,0)du\right];\quad \vartheta_{0n}=\frac{2}{\pi}\left[-\theta_4\psi_0(0,\mu_n)+\frac{2}{\pi}\sum_{j=1}^{\infty}\int_0^\infty\frac{\sin u}{u}G(hu)\psi_{j-1}(u,\mu_n)du\right];$$

$$\vartheta_{k0}=\frac{2}{\pi}\left[-\theta_4\psi_0(0,\mu_k)+\frac{2}{\pi}\sum_{j=1}^{\infty}\int_0^\infty\Big(\psi_{j-1}(u,\mu_k)G(hu)\psi_0(u,0)du\Big);\quad \vartheta_{00}=\frac{\theta_5\theta_3 RE}{\kappa l};\right. \qquad (54)$$

$$\vartheta_k=\frac{\theta_3\mu_k J_0^2(\mu_k)}{2\kappa R v_1}\left[\frac{l_2 v_2}{l_1 v_1}cth\left(\frac{\mu_k l}{v_2}\right)-cth\left(\frac{\mu_k l}{v_1}\right)\right];\quad \varpi_k=\frac{2}{\pi}\left[\psi_0(0,\mu_k)+\frac{2}{\pi}\sum_{j=1}^{\infty}\int_0^\infty\psi_{j-1}(u,\mu_k)G(hu)\psi_0(u,0)du\right];$$

$$\vartheta_{kn}=\frac{2}{\pi}\left[-\theta_4\psi_0(\mu_k,\mu_n)-\frac{2\theta_3 s_0 v_1 R\pi}{\kappa l}\sum_{m=1}^{\infty}\tau_{mn}\iota_{km}+\frac{2}{\pi}\sum_{j=1}^{\infty}\int_0^\infty\Big(\psi_0(u,\mu_n)G(hu)\psi_{j-1}(u,\mu_k)du\Big)\right];$$

where for the case of equal radicals $n_1=n_2$:

$$\vartheta_0=\frac{1}{\pi}\left[1+\frac{(m_2-1)R^2}{\theta_2}+\frac{2}{\pi}\sum_{j=1}^{\infty}\int_0^\infty\left(\psi_0(u,0)+\frac{2(m_2-1)R^2}{\theta_2}\psi_1(u,0)\right)G(hu)\psi_{j-1}(u,0)du\right];$$

$$\varpi_0=\frac{1}{\pi}\left[1-\frac{2}{\pi}\sum_{j=1}^{\infty}\int_0^\infty\psi_0(u,0)G(hu)\psi_{j-1}(u,0)du\right];\quad \vartheta_k=\frac{\mu_k J_0^2(\mu_k)}{2\theta_3\kappa R}\left[\frac{\mu_k}{Rv_1}E^{(k)}-sM^{(k)}\right];\qquad (55)$$

$$\vartheta_{0n}=\frac{1}{\pi}\left[\theta_4\psi_0(0,\mu_n)+\frac{(m_2-1)R^2}{2}b_1^{(n)}\psi_0(0,i\gamma_n v_1 R)+\frac{2}{\pi}\sum_{j=1}^{\infty}\int_0^\infty\Big(\theta_4\psi_0(u,\mu_n)+\right.$$

[43] Гузь А.Н., Рудницкий В.Б. Основы теории контактного взаимодействия упругих тел с начальными (остаточными) напряжениями. Хмельницький: вид. ПП Мельник, 2006. 710 с.

$$+\frac{(m_2-1)R^2}{2}b_1^{(n)}\psi_0(u,i\gamma_n v_1 R)\Big)G(hu)\psi_{j-1}(u,\mu_n)du\Bigg];$$

$$\vartheta_{00}=\frac{2E}{\kappa\theta_2\theta_3 lR};\quad \varpi_k=\frac{2}{\pi}\left[\psi_0(0,\mu_k)+\frac{2}{\pi}\sum_{j=1}^{\infty}\int_0^{\infty}\psi_0(u,\mu_k)G(hu)\psi_{j-1}(u,0)du\right];$$

$$\vartheta_{k0}=\frac{2}{\pi}\Bigg[\psi_0(0,\mu_k)+\frac{2(m_2-1)R^2}{\theta_2}\frac{\sin\mu_k}{\mu_k}+\frac{2}{\pi\varepsilon\theta_3}\sum_{j=1}^{\infty}\int_0^{\infty}\Big(\theta_4\psi_0(u,\mu_k)+$$

$$+\frac{(m_2-1)R^2}{2}b_1^{(k)}\psi_0(u,i\gamma_k v_1 R)\Big)G(hu)\psi_{j-1}(u,0)du\Bigg];$$

$$\vartheta_{kn}=-\frac{2}{\pi}\Bigg[\theta_4\psi_0(\mu_k,\mu_n)+\frac{(m_2-1)R^2}{2}b_1^{(k)}\psi_0(\mu_n,i\gamma_k v_1 R)+\frac{2}{\pi\varepsilon\theta_3}\sum_{j=1}^{\infty}\int_0^{\infty}\Big(\theta_4\psi_0(u,\mu_k)+$$

$$+\frac{(m_2-1)R^2}{2}b_1^{(k)}\psi_0(u,i\gamma_k v_1 R)\Big)G(hu)\psi_{j-1}(u,\mu_n)du+$$

$$+\frac{l}{2\kappa\theta_3}\sum_{m=1}^{\infty}\frac{\gamma_m^2 v_1^2 J_0(\mu_k)b_1^{(n)}(\gamma_m v_1 R I_1(\gamma_m v_1 R)+s_0(1-I_0(\gamma_m v_1 R)))}{\mu_k^2+\gamma_m^2 v_1^2 R^2}\Bigg].$$

where

$$\theta_5=(v_2+v_1 s)n_1 n_2((m_1 v_2^3+m_2 v_1^3)E)^{-1},$$

$$\psi_j(\eta,\mu_n)=\frac{2}{\pi}\eta\int_0^1\cos\eta t dt\int_0^{\infty}\frac{\psi_{j-1}(u,\mu_n)}{u}G(uh)\cos utdu.$$

Calculating functions (52) and coefficients (54) – (55), where the most of integrals are not finitely calculated, according to the complexity of G_i $(i=\overline{1,4})$. functions. So, starting the second approximation, subintegral functions decompose into lines by degrees h^{-i}, (i=1,..,7) which will allow to calculate the coefficients of the system (53) approximated, then the coefficients of the system (53) can be represented as:

for the case of equal radicals:

$$\vartheta_0=\pi^{-1}\Big\{\Big(1+\theta_2^{-1}(m_2-1)R^2\Big)\Big(1+2D_0(\pi h)^{-1}\Big)+2(\pi h^2)^{-1}\Big(4(m_2-1)R^2D_0^2(\pi\theta_2)^{-1}-D_2/3+2D_0^2\pi^{-1}\Big)+$$

$$+2(\pi h^3)^{-1}\Big(\Big(1+(m_2-1)R^2\theta_2^{-1}\Big)4D_0^3\pi^{-2}-2D_0D_2(3\pi)^{-1}-5(m_2-1)R^2D_2(12\theta_2)^{-1}\Big)-2D_0D_2(3\pi^3h^4)^{-1}(\pi+$$

$$+7\pi(m_2-1)R^2\theta_2^{-1}+4D_0)+2(3\pi h^5)^{-1}((48\theta_2+23(m_2-1)R^2)(120\theta_2)^{-1}D_4-D_2D_0^2\pi^{-2}(2\theta_2+9(m_2-$$

$$-1)R^2)\theta_2^{-1}+D_2^2(3\pi)^{-1})+2(3\pi h^6)^{-1}$$

$$\Big((55\theta_2+26(m_2-1)R^2)(5\theta_2)^{-1}D_4D_0+\Big(5(m_2-1)R^2\theta_2^{-1}+4D_0\pi^{-1}\Big)D_2^2\Big)+$$

$$+(3\pi h^7)^{-1}(16(m_2-1)R^2D_0^2D_4(5\pi^2\theta_2)^{-1}+4(m_2-1)R^2D_2^2D_0(\pi^2\theta_2)^{-1}-D_4D_0(30\pi)^{-1}-$$

$$-(192\theta_2+299(m_2-1)R^2)D_6(10080\theta_2)^{-1})\Big\},$$

$$\vartheta_k=\mu_k J_0^2(\mu_k)(2\theta_3\kappa R)^{-1}\Big[\mu_k(Rv_1)^{-1}E^{(k)}-sM^{(k)}\Big];\quad \vartheta_{00}=2E(\kappa\theta_2\theta_3 lR)^{-1};$$

$$\varpi_0 = \pi^{-1}\Big\{1 - 2\pi^{-1}\Big[D_0h^{-1} - (D_2/3 - 2D_0^2\pi^{-1})(h^{-2} + 2D_0(\pi h^3)^{-1}) - D_0D_2(3\pi h^4)^{-1}(1 + 4D_0\pi^{-1}) + 2(3h^5)^{-1}\times$$
$$\times(D_4/5 + D_2^2(6\pi)^{-1} - D_2D_0^2\pi^{-2}) +$$
$$+D_0(18\pi^2h^6)^{-1}(11D_4\pi + 4D_2^2) - (45h^7\pi^2)^{-1}(D_6\pi^2/7 + D_4D_2\pi/4 + D_4D_0^2/2)\Big]\Big\}$$

$$\vartheta_{0n} = \pi^{-1}\Big\{\theta_4\psi_0(0,\mu_n) + 0,5(m_2-1)R^2b_1^{(n)}\psi_0(0, i\gamma_n v_1R) +$$
$$+\pi^{-1}\Big[\Big(D_0h^{-1}r_1(\mu_n) + 4D_0^2(\pi h^2)^{-1}r_1^2(\mu_n)(1 + D_0(\pi h)^{-1})\Big)\Big(2\theta_4\pi^{-1}r_1(\mu_n) + (m_2-1)R^2\pi^{-1}b_1^{(n)}r_1(i\gamma_n v_1R)\Big) +$$
$$+D_2(2h^3)^{-1}\Big(2\theta_4\pi^{-1}r_8^{(1)}(\mu_n,\mu_n) + (m_2-1)R^2\pi^{-1}b_1^{(n)}r_8^{(1)}(\mu_n, i\gamma_n v_1R)\Big) +$$
$$+2D_0D_2(\pi^2h^4)^{-1}r_1(\mu_n)\Big(6\theta_4\mu_n^{-3}r_1(\mu_n)r_5(\mu_n) + (m_2-1)R^2b_1^{(n)}r_8^{(2)}(\mu_n, i\gamma_n v_1R)\Big) +$$
$$+(\pi h^5)^{-1}\Big(16\theta_4D_0^2D_2(\pi^2\mu_n^3)^{-1}r_1^3(\mu_n)r_5(\mu_n) + \theta_4D_4(6\mu_n^5)^{-1}(\mu_nr_1(\mu_n)r_6(\mu_n) + 3r_5^2(\mu_n)) +$$
$$+(m_2-1)R^2b_1^{(n)}\Big(D_4r_9^{(1)}(\mu_n, i\gamma_n v_1R)/24 + 2D_2D_0^2r_1^2(\mu_n)r_8^{(3)}(\mu_n, i\gamma_n v_1R)\pi^{-2}\Big)\Big) +$$
$$+(\pi^2h^6)^{-1}\Big(2\theta_4r_1(\mu_n)\mu_n^{-5}\Big(D_0D_4/12(2r_1(\mu_n)r_6(\mu_n)(1+2\mu_n) + 9r_5^2(\mu_n)) + 2D_2^2\mu_n^{-1}r_5^2(\mu_n)\Big) +$$
$$+(m_2-1)R^2b_1^{(n)}/6\Big(D_0D_4r_1(\mu_n)r_9^{(2)}(\mu_n, i\gamma_n v_1R) + 6D_2^2\mu_n^{-3}r_5(\mu_n)r_8^{(1)}(\mu_n, i\gamma_n v_1R)\Big)\Big) +$$
$$+(\pi h^7)^{-1}\Big(2\theta_4\mu_n^{-5}\Big(D_6(360\mu_n^3)^{-1}(15r_5(\mu_n)r_6(\mu_n) - \mu_nr_1(\mu_n)r_7(\mu_n)) +$$
$$+D_0\pi^{-2}r_1^2(\mu_n)\Big(5D_2^2r_5^2(\mu_n)\mu_n^{-1} + D_0D_4/3((1+\mu_n)r_1(\mu_n)r_6(\mu_n) + 3r_5^2(\mu_n))\Big)\Big) +$$
$$+(m_2-1)R^2b_1^{(n)}\Big(D_0^2D_4(6\pi^2)^{-1}r_1^2(\mu_n)r_9^{(3)}(\mu_n, i\gamma_n v_1R) + D_2^2D_0(\pi^2\mu_n^3)^{-1}r_1(\mu_n)r_5(\mu_n)(r_8^{(1)}(\mu_n, i\gamma_n v_1R) +$$
$$+r_8^{(2)}(\mu_n, i\gamma_n v_1R)) - D_6/720\,r_{10}(\mu_n, i\gamma_n v_1R)\Big)\Big)\Big]\Big\},$$

$$\varpi_k = 2\pi^{-1}\Big\{\psi_0(0,\mu_k) + 2\pi^{-1}\Big[D_0h^{-1}r_1(\mu_n) + 4D_0^2(\pi h^2)^{-1}r_1(\mu_n) + h^{-3}(4D_0^3\pi^{-2}r_1(\mu_n) + D_2(6\mu_n^3)^{-1}(3r_5(\mu_n) -$$
$$-\mu_k^3r_1(\mu_n))) + h^{-5}\mu_n^{-3}\Big(D_4(120\mu_n^2)^{-1}(5r_6(\mu_n) - 10\mu_n^2r_5(\mu_n) + \mu_n^5r_1(\mu_n)) + 2D_2D_0^2\pi^{-1}(r_5(\mu_n) - \mu_n^3r_1(\mu_n))\Big) +$$
$$+(90\pi\mu_n^5h^6)^{-1}\Big(3D_0D_4(5r_6(\mu_n) - 10\mu_n^2r_5(\mu_n) + 2\mu_n^5r_1(\mu_n)) - 10D_2^2\mu_n^2(3r_5(\mu_n) - \mu_n^3r_1(\mu_n))\Big) +$$
$$+(30\mu_n^5h^7)^{-1}\Big(D_0\pi^{-2}\Big(D_0D_4(5r_6(\mu_n) + 10\mu_n^2r_5(\mu_n) + 3\mu_n^5r_1(\mu_n)) - 10\mu_n^2D_2^2(2r_5(\mu_n) - \mu_n^3r_1(\mu_n))\Big) +$$
$$+D_6(168\mu_n^2)^{-1}(21\mu_n^4r_5(\mu_n) - 7r_7(\mu_n) - 35\mu_n^2r_6(\mu_n) - \mu_n^7r_1(\mu_n))\Big) +$$
$$+2D_0D_2(3\pi\mu_n^3h^4)^{-1}(3r_5(\mu_n) - 2\mu_k^3r_1(\mu_n)) + ..\Big\}\Big],$$

$$\vartheta_{k0} = 2\pi^{-1}\Big[\psi_0(0,\mu_k) + 2(m_2-1)\sin\mu_kR^2(\theta_2\mu_k)^{-1} + 2\theta_4(\pi\varepsilon\theta_3)^{-1}\Big\{D_0h^{-1}r_1(\mu_n) + 4D_0^2(\pi h^2)^{-1}r_1(\mu_n) +$$
$$+h^{-3}\Big(4D_0^3\pi^{-2}r_1(\mu_n) + D_2(6\mu_n^3)^{-1}(3r_5(\mu_n) - \mu_k^3r_1(\mu_n))\Big) + 2D_0D_2(3\pi\mu_n^3h^4)^{-1}(3r_5(\mu_n) - 2\mu_k^3r_1(\mu_n)) +$$
$$+(h^5\mu_n^3)^{-1}\Big(D_4(120\mu_n^2)^{-1}(5r_6(\mu_n) - 10\mu_n^2r_5(\mu_n) + \mu_n^5r_1(\mu_n)) + 2D_2D_0^2\pi^{-1}(r_5(\mu_n) - \mu_n^3r_1(\mu_n))\Big) +$$
$$+(90\pi\mu_n^5h^6)^{-1}\Big(3D_0D_4(5r_6(\mu_n) - 10\mu_n^2r_5(\mu_n) + 2\mu_n^5r_1(\mu_n)) - 10D_2^2\mu_n^2(3r_5(\mu_n) - \mu_n^3r_1(\mu_n))\Big) +$$
$$+(30\mu_n^5h^7)^{-1}\Big(D_0\pi^{-2}\Big(D_0D_4(5r_6(\mu_n) + 10\mu_n^2r_5(\mu_n) + 3\mu_n^5r_1(\mu_n)) - 10\mu_n^2D_2^2(2r_5(\mu_n) - \mu_n^3r_1(\mu_n))\Big) +$$
$$+D_6(168\mu_n^2)^{-1}(21\mu_n^4r_5(\mu_n) - 7r_7(\mu_n) - 35\mu_n^2r_6(\mu_n) - \mu_n^7r_1(\mu_n))\Big)\Big\} +$$
$$+(m_2-1)R^2(\pi\varepsilon\theta_3)^{-1}b_1^{(k)}\Big\{D_0h^{-1}r_1(i\gamma_kv_1R) +$$

$$+4D_0^2(\pi h^2)^{-1}r_1(i\gamma_k v_1 R) + h^{-3}\left(4D_0^3\pi^{-2}r_1(i\gamma_k v_1 R) + D_2(6\mu_n^3)^{-1}(3r_5(i\gamma_k v_1 R) - \mu_k^3 r_1(i\gamma_k v_1 R))\right) + \quad (56)$$

$$+2D_0 D_2(3\pi\mu_n^3 h^4)^{-1}(3r_5(i\gamma_k v_1 R) - 2\mu_k^3 r_1(i\gamma_k v_1 R)) + (h^5\mu_n^3)^{-1}\left(D_4(120\mu_n^2)^{-1}(5r_6(\mu_n) - 10\mu_n^2 r_5(\mu_n) +\right.$$

$$\left.+\mu_n^5 r_1(\mu_n)) + 2D_2 D_0^2\pi^{-1}(r_5(\mu_n) - \mu_n^3 r_1(\mu_n))\right) + (90\pi\mu_n^5 h^6)^{-1}(3D_0 D_4(5r_6(\mu_n) - 10\mu_n^2 r_5(\mu_n) +$$

$$+2\mu_n^5 r_1(\mu_n)) - 10D_2^2\mu_n^2(3r_5(\mu_n) - \mu_n^3 r_1(\mu_n))) + (30\mu_n^5 h^7)^{-1}\left(D_0\pi^{-2}(D_0 D_4(5r_6(\mu_n) + 10\mu_n^2 r_5(\mu_n) +\right.$$

$$+3\mu_n^5 r_1(\mu_n)) - 10\mu_n^2 D_2^2(2r_5(\mu_n) - \mu_n^3 r_1(\mu_n))) + D_6(168\mu_n^2)^{-1}(21\mu_n^4 r_5(\mu_n) - 7r_7(\mu_n) -$$

$$\left.\left.-35\mu_n^2 r_6(\mu_n) - \mu_n^7 r_1(\mu_n)))\right\}\right],$$

$$\vartheta_{kn} = -2\pi^{-1}\left[\theta_4\psi_0(\mu_k,\mu_n) + l(2\kappa\theta_3)^{-1}\sum_{m=1}^{\infty}\gamma_m^2 v_1^2 J_0(\mu_k)b_1^{(n)}(\gamma_m v_1 R I_1(\gamma_m v_1 R) + s_0(1 - I_0(\gamma_m v_1 R)))(\mu_k^2 + \gamma_m^2 v_1^2 R^2)^{-1} +\right.$$

$$+0,5(m_2 - 1)R^2 b_1^{(k)}\psi_0(\mu_n, i\gamma_k v_1 R) + 2\theta_4(\pi\varepsilon\theta_3)^{-1}\left\{D_0 h^{-1}r_1(\mu_k)r_1(\mu_n) + 4D_0^2(\pi h^2)^{-1}r_1(\mu_n)r_1^2(\mu_k) +\right.$$

$$+h^{-3}\left(4D_0^3\pi^{-2}r_1(\mu_n)r_1^2(\mu_k) + 0,5D_2 r_8^{(1)}(\mu_k,\mu_n)\right) + 2D_0 D_2(\pi h^4)^{-1}r_1(\mu_k)r_8^{(2)}(\mu_k,\mu_n) +$$

$$+h^{-5}\left(D_4/24\, r_9^{(1)}(\mu_k,\mu_n) + 2D_2 D_0^2\pi^{-2}r_1^2(\mu_k)r_8^{(3)}(\mu_k,\mu_n)\right) +$$

$$+(6\pi h^6)^{-1}\left(D_0 D_4 r_1(\mu_k)r_9^{(2)}(\mu_k,\mu_n) + 6D_2^2\mu_k^{-3}r_5(\mu_k)r_8^{(1)}(\mu_k,\mu_n)\right) +$$

$$+h^{-7}\left(2D_2^2 D_0\pi^{-2}\mu_k^{-3}r_1(\mu_k)r_5(\mu_k)r_8^{(1,5)}(\mu_k,\mu_n) + +\right.$$

$$\left.\left.D_0^2 D_4(6\pi^2\mu_k^5)^{-1}r_9^{(3)}(\mu_k,\mu_n)r_1^2(\mu_k) - D_6/720\, r_{10}(\mu_k,\mu_n)\right)\right\} +$$

$$+(m_2 - 1)R^2 b_1^{(k)}(\pi\varepsilon\theta_3)^{-1}\left\{D_0 h^{-1}r_1(\mu_n)r_1(i\gamma_k v_1 R) + 4D_0^2(\pi h^2)^{-1}r_1(i\gamma_k v_1 R)r_1^2(\mu_n) +\right.$$

$$+h^{-3}\left(4D_0^3\pi^{-2}r_1(i\gamma_k v_1 R)r_1^2(\mu_n) + 0,5D_2 r_8^{(1)}(i\gamma_k v_1 R,\mu_n)\right) + 2D_0 D_2(\pi h^4)^{-1}r_1(\mu_n)r_8^{(2)}(i\gamma_k v_1 R,\mu_n) +$$

$$+h^{-5}\left(D_4/24\, r_9^{(1)}(i\gamma_k v_1 R,\mu_n) + 2D_2 D_0^2\pi^{-2}r_1^2(\mu_n)r_8^{(3)}(i\gamma_k v_1 R,\mu_n)\right) +$$

$$+(6\pi h^6)^{-1}\left(D_0 D_4 r_1(\mu_n)r_9^{(2)}(i\gamma_k v_1 R,\mu_n) + 6D_2^2\mu_k^{-3}r_5(\mu_n)r_8^{(1)}(i\gamma_k v_1 R,\mu_n)\right) + h^{-7}\left(2D_2^2 D_0(\pi^2\mu_k^3)^{-1}r_1(\mu_n)\times\right.$$

$$\left.\left.\left.\times r_8^{(1,5)}(i\gamma_k v_1 R,\mu_n)r_5(\mu_n) + D_0^2 D_4(6\pi^2\mu_k^5)^{-1}r_9^{(3)}(i\gamma_k v_1 R,\mu_n)r_1^2(\mu_n) - D_6/720\, r_{10}(i\gamma_k v_1 R,\mu_n)\right)\right\}\right]$$

for the case of the different radicals:

$$\vartheta_0 = \varpi_0 = 2\pi^{-1}\left[1 + 2\pi^{-1}\left\{\begin{matrix} D_0 h^{-1} - h^{-2}\left(D_2/3 - 2D_0^2\pi^{-1}\right)\left(h + 2D_0\pi^{-1}\right) \\ -D_0 D_2(3\pi^2 h^4)^{-1}(\pi + 4D_0) + 2(3h^5)^{-1}(D_4/5 + \end{matrix}\right.\right.$$

$$\left.\left.+D_2^2(6\pi)^{-1} - D_2 D_0^2\pi^{-2}) + \begin{matrix} D_0(18\pi h^6)^{-1}(11D_4 + 4D_2^2\pi^{-1}) - (15h^7)^{-1} \\ \left(D_6/21 + D_2 D_4(12\pi)^{-1} - 19D_4 D_0^2(6\pi^2)^{-1}\right) + .. \end{matrix}\right\}\right];$$

$$\vartheta_{0n} = 2\pi^{-1}\left[-\theta_4\psi_0(0,\mu_n) + 2\pi^{-1}\left\{\begin{matrix} D_0 h^{-1}r_1(\mu_n) + 4D_0^2(\pi h^2)^{-1}r_1^2(\mu_n) + \\ +h^{-3}\left(4D_0^3\pi^{-2}r_1^3(\mu_n) - D_2(3\mu_n^3)^{-1}r_2(\mu_n)\right) + \end{matrix}\right.\right.$$

$$+2D_0 D_2(3\pi\mu_n^3 h^4)^{-1}r_1(\mu_n)(3r_5(\mu_n) - 2r_2(\mu_n)) + (3h^5\mu_n^3)^{-1}\begin{pmatrix} D_4\mu_n^{-2}r_3(\mu_n) + \\ +2D_2 D_0^2\pi^{-1}r_1^2(\mu_n)(3r_5(\mu_n) - r_2(\mu_n)) \end{pmatrix} +$$

$$+2(3\pi\mu_n^5 h^6)^{-1}\left(D_0 D_4/8\, r_1(\mu_n)(16r_3(\mu_n) + 3r_6(\mu_n)) - D_2^2\mu_n^{-1}r_2(\mu_n)r_5(\mu_n)\right) +$$

$$+(3\mu_n^5 h^7)^{-1}\left(D_0\pi^{-2}r_1(\mu_n)\left(4D_0 D_4/3\, r_1(\mu_n)(3r_3(\mu_n) + r_6(\mu_n)) + D_2^2\mu_n^{-1}r_5(\mu_n)(3r_5(\mu_n) - 4r_2(\mu_n))\right) -\right.$$

$$\left.\left.\left. - D_6(168\mu_n^2)^{-1}(21\mu_n^4 r_5(\mu_n) - 7r_7(\mu_n) - 35\mu_n^2 r_6(\mu_n) - \mu_n^7 r_1(\mu_n))\right) + ..\right\}\right], \qquad (57)$$

$$\vartheta_k = \theta_3\mu_k J_0^2(\mu_k)(2\kappa R v_1)^{-1}\left[l_2 v_2 (l_1 v_1)^{-1} cth(\mu_k l v_2^{-1}) - cth(\mu_k l v_1^{-1})\right]; \quad \vartheta_{00} = \theta_5\theta_3 RE(\kappa l)^{-1};$$

$$\varpi_k = 2\pi^{-1}\left[\psi_0(0,\mu_k) + 2\pi^{-1}\begin{cases} D_0 h^{-1} r_1(\mu_k) + 4D_0^2(\pi h^2)^{-1} r_1^2(\mu_k) + \\ +h^{-3}\left(4D_0^3\pi^{-2} r_1^3(\mu_k) - D_2(3\mu_k^3)^{-1} r_2(\mu_k)\right) + \end{cases}\right.$$

$$+2D_0 D_2(3\pi\mu_k^3 h^4)^{-1} r_1(\mu_k)(3r_5(\mu_k) - 2r_2(\mu_k)) + (3h^5\mu_k^3)^{-1}$$

$$\left(D_4\mu_k^{-2} r_3(\mu_k) + 2D_2 D_0^2\pi^{-1} r_1^2(\mu_k)(3r_5(\mu_k) - r_2(\mu_k))\right) +$$

$$+2(3\pi\mu_k^5 h^6)^{-1}\left(D_0 D_4/8\, r_1(\mu_k)(16 r_3(\mu_k) + 3r_6(\mu_k)) - D_2^2\mu_k^{-1} r_2(\mu_k) r_5(\mu_k)\right) + (3\mu_k^5 h^7)^{-1} \times$$

$$\left.\times\left(D_0\pi^{-2} r_1(\mu_k)\begin{pmatrix} 4D_0 D_4/3\, r_1(\mu_k)(3r_3(\mu_k) + r_6(\mu_k)) + \\ +D_2^2\mu_k^{-1} r_5(\mu_k)(3r_5(\mu_k) - 4r_2(\mu_k)) \end{pmatrix} - D_6(5\mu_k^2)^{-1} r_4(\mu_k)\right) + ..\}\right]$$

$$\vartheta_{k0} = 2\pi^{-1}\left[-\theta_4\psi_0(0,\mu_k) + 2\pi^{-1}\begin{cases} D_0 h^{-1} r_1(\mu_k) + 4D_0^2(\pi h^2)^{-1} r_1^2(\mu_k) + \\ +h^{-3}\left(4D_0^3\pi^{-2} r_1^3(\mu_k) - D_2(3\mu_k^3)^{-1} r_2(\mu_k)\right) + \end{cases}\right.$$

$$+2D_0 D_2(3\pi\mu_k^3 h^4)^{-1} r_1(\mu_k)(3r_5(\mu_k) - 2r_2(\mu_k)) + (3h^5\mu_k^3)^{-1}$$

$$\left(D_4\mu_k^{-2} r_3(\mu_k) + 2D_2 D_0^2\pi^{-1} r_1^2(\mu_k)(3r_5(\mu_k) - r_2(\mu_k))\right) +$$

$$+2D_0 D_2(3\pi\mu_k^3 h^4)^{-1} r_1(\mu_k)(3r_5(\mu_k) - 2r_2(\mu_k)) +$$

$$+(3h^5\mu_k^3)^{-1}\left(D_4\mu_k^{-2} r_3(\mu_k) + 2D_2 D_0^2\pi^{-1} r_1^2(\mu_k)(3r_5(\mu_k) - r_2(\mu_k))\right) +$$

$$+2(3\pi\mu_k^5 h^6)^{-1}\left(D_0 D_4/8\, r_1(\mu_k)(16 r_3(\mu_k) + 3r_6(\mu_k)) - D_2^2\mu_k^{-1} r_2(\mu_k) r_5(\mu_k)\right) + (3\mu_k^5 h^7)^{-1} \times$$

$$\left.\times\left(D_0\pi^{-2} r_1(\mu_k)\begin{pmatrix} 4D_0 D_4/3\, r_1(\mu_k)(3r_3(\mu_k) + r_6(\mu_k)) + \\ +D_2^2\mu_k^{-1} r_5(\mu_k)(3r_5(\mu_k) - 4r_2(\mu_k)) \end{pmatrix} - D_6(5\mu_k^2)^{-1} r_4(\mu_k)\right) + ..\}\right]$$

$$\vartheta_{kn} = 2\pi^{-1}\left[-\theta_4\psi_0(\mu_n,\mu_k) - 2\theta_3 s_0 v_1 R\pi(\kappa l)^{-1}\sum_{m=1}^{\infty}\tau_{mn}\iota_{km} + \right.$$

$$+2\pi^{-1}\left\{D_0 h^{-1} r_1(\mu_k) r_1(\mu_n) + 4D_0^2(\pi h^2)^{-1} r_1(\mu_n) r_1^2(\mu_k) + h^{-3}\left(4D_0^3\pi^{-2} r_1(\mu_n) r_1^2(\mu_k) + 0,5 D_2 r_8^{(1)}(\mu_k,\mu_n)\right) + \right.$$

$$+2D_0 D_2(\pi h^4)^{-1} r_1(\mu_k) r_8^{(2)}(\mu_k,\mu_n) + h^{-5}\left(D_4/24\, r_9^{(1)}(\mu_k,\mu_n) + 2D_2 D_0^2\pi^{-2} r_1^2(\mu_k) r_8^{(3)}(\mu_k,\mu_n)\right) +$$

$$+(6\pi h^6)^{-1}\left(D_0 D_4 r_1(\mu_k) r_9^{(2)}(\mu_k,\mu_n) + 6D_2^2\mu_k^{-3} r_5(\mu_k) r_8^{(1)}(\mu_k,\mu_n)\right) +$$

$$+h^{-7}\left(2D_2^2 D_0(\pi^2\mu_k^3)^{-1} r_1(\mu_k) r_5(\mu_k) r_8^{(1,5)}(\mu_k,\mu_n) + \right.$$

$$\left.\left.\left. + D_0^2 D_4(6\pi^2\mu_k^5)^{-1} r_9^{(3)}(\mu_k,\mu_n) r_1^2(\mu_k) - D_6/720\, r_{10}(\mu_k,\mu_n)\right) + ..\right\}\right].$$

where

$$r_1(\mu_k) = \mu_k^{-1}\sin\mu_k; \quad r_2(\mu_k) = 3\mu_k\cos\mu_k + (2\mu_k^2 - 3)\sin\mu_k; \quad r_3(\mu_k) = 0,4\mu_k^4\sin\mu_k + \mu_k^3\cos\mu_k - r_2(\mu_k);$$

$$r_4(\mu_k) = 2/3\,\mu_k^5(0,4\mu_k\sin\mu_k + \cos\mu_k) - 5\mu_k(\mu_k^2 - 3)\cos\mu_k - (2\mu_k^4 - 10\mu_k^2 + 15)\sin\mu_k;$$

$$r_5(\mu_k) = (2 - \mu_k^2)\sin\mu_k - 2\mu_k\cos\mu_k; \quad r_6(\mu_k) = (\mu_k^4 - 12\mu_k^2 + 1)\sin\mu_k + (4\mu_k^2 - 1)\mu_k\cos\mu_k;$$

$$r_7(\mu_k) = (\mu_k^6 - 30\mu_k^4 + 360\mu_k^2 - 720)\sin\mu_k + 6\mu_k(\mu_k^4 - 20\mu_k^2 + 120)\cos\mu_k.$$

$$r_8^{(m)}(\mu_k,\mu_n) = \frac{m r_5(\mu_k) r_1(\mu_n)}{\mu_k^3} + \frac{r_5(\mu_n) r_1(\mu_k)}{\mu_n^3}; \quad r_9^{(m)}(\mu_k,\mu_n) = \frac{r_1(\mu_k) r_6(\mu_n)}{\mu_k^5} + \frac{6 r_5(\mu_n) r_5(\mu_k)}{\mu_n^3\mu_k^3} + \frac{m r_6(\mu_k) r_1(\mu_n)}{\mu_n^5};$$

$$r_{10}(\mu_k,\mu_n) = r_1(\mu_k) r_7(\mu_n)\mu_n^{-7} - 15 r_6(\mu_n) r_5(\mu_k)\mu_n^{-5}\mu_k^{-3} - 15 r_5(\mu_n) r_6(\mu_k)\mu_n^{-3}\mu_k^{-5} + r_7(\mu_k) r_1(\mu_n)\mu_k^{-7}.$$

$$D_n = \int_0^\infty t^n G(t)dt. \qquad (58)$$

Find (58), which are the coefficients of the deviding (56). Thus (58) can not be expressed by the elementary functions, then missing some layouts, will be achieved

$$D_n = \sum_{i=1}^{m} \frac{a_i}{(k_i)^{n+1}} \Gamma(\mathrm{n+1}). \qquad (59)$$

where k_i, a_i = *const*, *i*=0,1,2,.. .

4. Numerical analysis and numerical results

Determining unknown constants χ_i $(i = 0,1,2,..)$ from the system (53), it is possible to calculate the stress and strain state in both the elastic punch and the layer by formulas (35) – (40).

As a result, the solution is represented as rows through the infinite system of constants defined from the system of quasiregular linear algebraic equations. Moreover, in the system (53), the coefficients ϑ_k and ϑ_{kn} depend on the structure of the elastic potential, the height of the elastic punch *H*, and the thickness of the previously stressed layer, and the free members depend on the radicals n_1, n_2 .

Take into account the asymptotic representations for Bessel functions, μ_k quantities, ψ and integral limitations $\psi(\mu_k,\ \mu_n)$, the system (53) is quasiregular if $\lambda_1 > \lambda_{кр}$, if the condition is realized

$$C_{44} l_1 (1+m_1)(s-s_0)(m_1(s_0-s_1))^{-1} < \begin{cases} 0,36E(1-v^2)^{-1}, & \text{для стисливих тіл;} \\ 0,48E, & \text{для нестисливих тіл,} \end{cases}$$

Numerically, the quasi-regularity of the system (53) confirms Table. 1, formed for the first eight values of the coefficients of the system, written in the form

$$\chi_k = -\sum_{n=0}^{\infty} \vartheta_{kn} / \vartheta_k \cdot \chi_n + \varpi_k / \vartheta_k \quad (k = 0,1,2,..)$$

The research concludes the numerical solution of the system (53) for Treloar potentials (Neoguki solid) and harmonic potentials if the following parameter values: *k*=*n*=32; $v = v_1 = 0,5$; *l*=10; $\lambda_1 = 0,7$; 0,8; 0,9; 1; 1,1;

1,2; E=3,92. The algorithm is based on the reduction method and implemented as a program in the Maple package[44].

Table 1

Coefficients of quasiregular system of linear algebraic equations

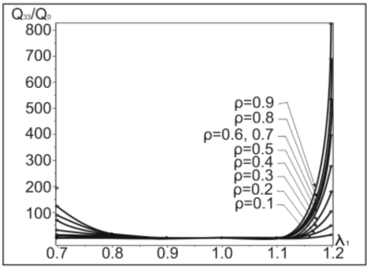
№ k	$-\vartheta_{kn}/\vartheta_k$								ϖ_k/ϑ_k
	1	2	3	4	5	6	7	8	
1	0.67817	0.67816	-0.49398	-0.49397	0.36871	0.36870	-0.27927	-0.27926	$3.62\cdot10^{-5}$
2	0.67814	0.67813	-0.49393	-0.49392	0.36865	0.36864	0.27922	0.27921	$-3.39\cdot10^{-6}$
3	0.67813	0.67812	-0.49392	-0.49391	0.36864	0.36863	-0.27920	-0.27919	$-1.26\cdot10^{-6}$
4	0.67810	0.67809	-0.49389	-0.49388	0.36861	0.36860	-0.27918	-0.27916	$-1.09\cdot10^{-6}$
5	0.67808	0.67808	-0.49384	-0.49383	0.36854	0.36853	-0.27911	-0.27910	$6.72\cdot10^{-7}$
6	0.67808	0.67807	-0.49383	-0.49382	0.36854	0.36853	-0.27910	-0.27909	$7.58\cdot10^{-8}$
7	0.67807	0.67806	-0.49382	-0.49381	0.36852	0.36851	-0.27908	-0.27907	$2.43\cdot10^{-8}$
8	0.67789	0.67788	-0.49352	-0.49351	0.36819	0.36818	-0.27875	-0.27874	$-8.97\cdot10^{-9}$

The influence of initial stresses to the law of dividing contact stresses and displacements for the problem of the elastic cylindrical punch pressure on the layer with initial (residual) stresses in the case of harmonic potential is depicted in Fig. 3, 5, and in the case of Treloar potential in Fig. 2, 4 and 6. Moreover, Fig. 5 presents tangential stresses that are most concentrated near the contact zone.

The research concludes the convergence of numerical rows from the (26) – (31). So, for the most rows found majors. Convergence of some rows were quite difficult to prove analytically, but from the numerical results it turned out that it is ensured by a monotonous decline of constant $\chi_i \quad (i=0,1,2,..)$ and $|J_0(\mu_k\rho)|$. But some rows are the part of the stresses of the cylindrical punch expressions (37) – (38) in the points of change boundary conditions change were found divergent, (because $\mu_k\cdot\chi_k\cdot J_0(\mu_k\rho)\to\infty$ if $k\to\infty$) but it is coordinates with the research[45].

[44] Ярецька Н. О. А. с. KNDS_CS_PZN. Комп'ютерна програма «Розрахунок компонентів напружено-деформованого стану для осесиметричної статичної задачі про тиск пружного циліндричного штампа на пружний шар з початковими (залишковими) напруженнями» № 54576; заявл. 05.05.2014; опубл. 01.09.2014, Бюл. № 34, 10 с.

[45] Гузь А.Н., Рудницкий В.Б. Основы теории контактного взаимодействия упругих тел с начальными (остаточными) напряжениями. Хмельницький: вид. ПП Мельник, 2006. 710 с.



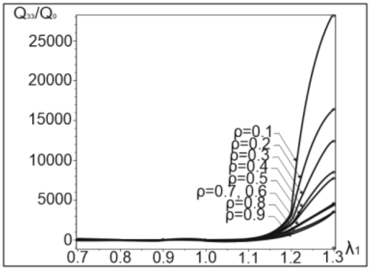
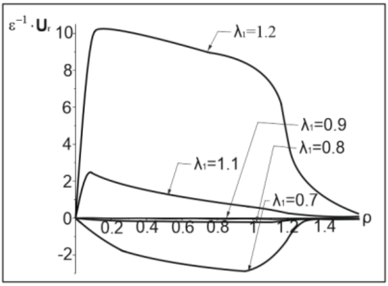


Fig. 2. Effect of initial stresses on the normal law of distribution (Treloar potential)

Fig. 3. Effect of initial stresses on normal distribution law (harmonic potential)



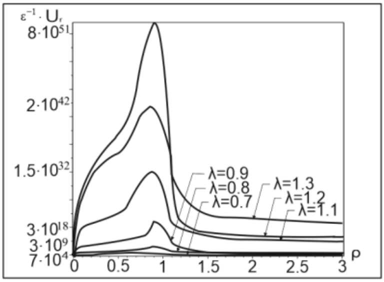


Fig. 4. Displacement Ur in the elastic layer (Treloar potential)

Fig. 5. Move Ur in the elastic layer (harmonic potential)

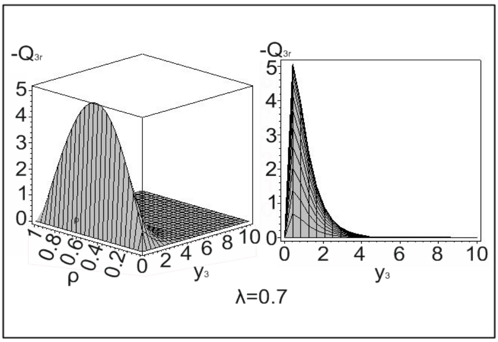


Fig. 6. Tangential stresses (Treloar potential)

Comparing the components of the stress and strain state of the solids with the initial stresses with the corresponding expressions for an isotropic solid without initial stresses, if $z_i = 0$ was gotten the equation:

$$U_3(r,0) = \mathrm{k} \cdot U_3^0(r,0),\ Q_{33}(r,0) = \mathrm{k}_s \cdot Q_{33}^0(r,0), \qquad (60)$$

where $U_3(r,0)$, $Q_{33}(r,0)$ – displacement and tension under the punch, squeezed into the layer with initial stresses; $U_3^0(r,0)$, $Q_{33}^0(r,0)$ – displacement and tension under the punch, squeezed into the layer without initial tension; k, k_s – coefficients that reflect the effect of initial stresses on the contact stresses and displacement of the elastic cylinder and layer.

The dependence of change coefficients k, ks of equations (60) is presented in Table 2. What shows that when the elongation coefficient approaches to the surface instability values of the material, the movement increases indefinitely, and the stresses go to zero.

Table 2

Variation dependence of *k*, k_s coefficients

λ_1	Potential Barteneva-Khazanovych		The potential of Treloar		Harmonic potential	
	k	k_s	k	k_s	k	k_s
0,5951	–	–	–	–	∞	0
0,6661	–	–	∞	0	1,7391	0,2332
0,6934	∞	0	4,1602	0,2090	1,5396	0,3128
0,7	19,7913	44,3841	3,4487	2,9543	1,5061	4,7907
0,8	1,7088	2,6107	1,3285	1,2423	1,2446	2,3223
0,9	1,1653	1,3597	1,0774	1,0376	1,1166	1,4539
1,1	0,9328	0,8847	0,9583	1,0218	0,8533	0,7142
1,2	0,9048	0,8778	0,9176	1,0699	0,6306	0,5132
1,3	0,8961	0,9327	0,8687	1,1269	0,2329	0,3609

Checking the results on the reference tasks are represented numerically in Table. 3, comparing the numerical values of force *P*, acting on the upper end of the punch, if the given values of initial stresses and thickness of the layer *h* with the case without initial stresses (allocated in bold).

Table 3

Numerical values of force $P/\varepsilon R$

	λ_1 / h	0,7	0,8	0,9	**1**	1,1	1,2
The potential of Treloar	1,6	1,4082	1,2487	1,2974	**1,2315**	1,2978	1,2043
	4	1,4025	1,2456	1,2945	**1,2296**	1,2653	1,2022

Minimum values of layer thickness *h* are presented in Table. 4, in the case of harmonic potential

Table 4

Minimum layer thickness values *h*

λ_1	0,7	0,8	0,9	1	1,1	1,2	1,3
h	1,54	1,25	1,02	0,83	0,67	0,54	0,42
t	1,49	1,27	1,08	0,83	0,65	0,51	0,41

For comparison, the values of layer thickness *t* are given when the cylinder does not have the initial stresses. From Table. 4 it is visible that the initial stresses affect the method of consecutive approximations.

CONCLUSIONS

Thus, take into account the given results of mathematical modeling and the conducted research for potentials corresponding to the equal and different radicals of the defining equation (3), the effect of initial stresses on the stress-deformed state of the elastic cylinder squeezed into the elastic layer and the base is that:

1. Initial stresses during compression lead to the decrease in the stress force in the cylindrical punch and the layer, and during the stretching – to increase them, in the case of displacement, everything happens vice versa. That is, the presence of the pre-stressed state during the contact interaction of elastic solids allows to adjust contact stresses and displacements calculating the strength of parts of machines and structures. Moreover, for the contact stresses, the initial stresses are dangerous in the case of stretching, and for moving – in the case of compression.
2. The greatest impact of initial stresses is noted on the side surface of the punch.
3. The thickness of the layer does not affect the nature of the initial stresses, but only affects their values.
4. More significantly, quantitatively, the initial stresses act in high-elastic materials compared to more rigid, but qualitatively their impact is preserved.
5. The dangerous situation is if the initial stresses approach the values of surface instability, as contact stresses and movements dramatically change their values.

The influence of initial (residual) stresses found in the study is essential for compressive and incompressible solids and should be taken into account if calculating the reliability and strength of materials, constructions, structures and equipment. This is confirmed by the obtained analytical, graphical and numerical results, which makes it possible to use them in engineering calculations.

SUMMARY

The article deals with the coaxial mixed type task of measuring pressure of an elastic cylinder die upon a layer with initial stresses within the framework of linearized theory of elasticity. In general, the research was carried out for the theory of large (finite) initial deformations and two variants of the theory of small initial deformations with the elastic potential having arbitrary form. Investigated the question about the influence of initial stresses on the distribution of the contact forces in elastic layer and punch.

REFERENCES

1. Повідомлення НАН України. URL: http://www.nas.gov.ua/UA/Messages/Pages/View.aspx?MessageID=7263

2. Babich. S. Yu., Dikhtyaruk N. N. Load transfer from an infinite inhomogeneous stringer to an elastic strip clamped by one face with initial stresses. *International Applied Mechanics*. 2020. Vol. 56. №6. P. 346–356. https://doi.org/10.1007/s10778-021-01047-9

3. *Guz A. N.* Nonclassical Problems of Fracture/Failure Mechanics: On the Occasion of the 50th Anniversary of Research (Review). III. *International Applied Mechanics*. 2019. Vol. 55. № 4. Pp. 343–415. https://doi.org/10.1007/s10778-019-00960-4

4. Гузь А.Н., Бабич С.Ю., Глухов Ю.П. Смешанные задачи для упругого основания с начальными напряжениями. Германия: Saarbrücken LAPLAMBERT Academic Publishing, 2015. 468 с.

5. Гузь А.Н., Рудницкий В.Б. Основы теории контактного взаимодействия упругих тел с начальными (остаточными) напряжениями. Хмельницький: вид. ПП Мельник, 2006. 710 с.

6. Rushchitsky J.J. Auxetic metamaterials from the position of mechanics: linear and nonlinear models. *Dopov. Nac. akad. nauk Ukr*. 2018. № 7. P. 46–58. doi: https://doi.org/10.15407/dopovidi2018.07.046.

7. Kappus R. Zur Elastizctatstheorie tndlicher Verschiebunger. *ZAMM*. 1939. Vol. 19. № 5. P. 271–15.

8. Biot M. A. Mechanics of incremental deformations. New York: John Willey and Sons. 1965. 126 p.

9. Гузь О.М., Бабич С.Ю., Рудницький В.Б. Контактна взаємодія тіл з початковими напруженнями: Навчальний посібник. Київ: Вища школа, 1995. 304 с.

10. Guz, O.M., Babych, S.Y., Glukhov, A.Y. Axisymmetric Waves in Prestressed Highly Elastic Composite Material. Long Wave Approximation. *International Applied Mechanics*. 2021. Vol. 57. № 2. P. 134–147. https://doi.org/10.1007/s10778-021-01068-4

11. Babych, S.Y., Glukhov, Y.P. On One Dynamic Problem for a Multilayer Half-Space with Initial Stresses. *International Applied Mechanics.* 2021. Vol. 57. №1. P. 43–52. https://doi.org/10.1007/s10778-021-01061-x

12. Yaretskaya N. A. Three-Dimensional Contact Problem for an Elastic Layer and a Cylindrical Punch with Prestresses. *International Applied Mechanics*. 2014. Vol. 50, № 4. P. 378–388. https://doi.org/10.1007/s10778-014-0641-y

13. Гузь А. Н., Бабич С.Ю., Рудницкий В.Б. Контактное взаимодействие упругих тел с начальными (остаточными) напряжениями. *Развитие идей Л. А. Галина в механике*. Москва : Ижевск. Институт компьютерных исследований. 2013. 480 с.

14. Guz A. N., Guz I.A The stability of the interface between two bodies compressed along interface cracks. 3. Exact solutions for the combined case of equal and unequal roots. *Int. Appl. Mech*. 2000. Vol. 36. № 6. P. 759–768. https://doi.org/10.1007/BF02681983.

15. Neuber H. Theorie der elastischen Stabilitat bei nichtlinearer Vorverfomung. *Acta mech*. 1965. Vol. 1. № 3. P. 112–32.

16. Guz A.N. Elastic waves in bodies with initial (residual) stresses. *Int. Appl: Mech*. 2002. Vol. 38. № 1. P. 23–59. https://doi.org/10.1023/A:1015379824503.

17. Гузь А.Н., Бабич С.Ю., Глухов Ю.П. Статика и динамика упругих оснований с начальными (остаточными) напряжениями: Монография. Кременчук: «Press – Line», 2007. 795 с.

18. Guz A.N., Babich S.Y., Rudnitskii V.B. Contact problems for elastic bodies with initial stresses: Focus on Ukrainian research. *Int. Appl. Mech. Rew*. 1998. Vol. 51. №5. P. 343–371. https://doi.org/10.1115/1.3099009

19. Guz A. N., Guz I.A. Mixed plane problems in lianearized solid mechanics. Exact solutions. *Int. Appl. Mech*. 2004; Vol. 40. № 1. C. 1–29. https://doi.org/10.1023/B:INAM.0000023808.08859.48.

20. Guz A. N. Establishing the fundamentals of the theory of stability of mine workings. *Int. Appl. Mech*. 2003. Vol. 39. № 1. P. 20–48. https://doi.org/10.1023/A:1023659931802.

21. Babych, S.Y., Yarets'ka, N.O. Contact Problem for an Elastic Ring Punch and a Half-Space with Initial (Residual) Stresses. *Int Appl Mech.* 2021. Vol. 57. № 3. P. 297–305. https://doi.org/10.1007/s10778-021-01081-7.

22. Развитие теории контактных задач в СССР / под ред. Л. А. Галина. Москва : Наука, 1976. 494 с.

23. Aleksandrov V. M. , Arutyunyan N. Ky. Contact problems for prestressed deformed bodies. *Soviet Applied Mechanics.* 1984. Vol. 20. №3. P. 209–215. https://doi.org/10.1007/BF00883134.

24. Yaretskaya N. A. Contact Problem for the Rigid Ring Stamp and the Half-Space with Initial (Residual) Stresses. *International Applied Mechanics*. 2018. Vol. 54, № 5. P. 539–543. https://doi.org/10.1007/s10778-018-0906-y.

25. Тихонов А. Н., Самарский А. А. Уравнения математической физики. Издание 3, испр. и доп. Москва : Наука, 1966. 724 с.

26. Ватсон Г. Н. Теория бесселевых функций. Часть 1. Москва, 1949. 798 с.

27. Бобик О. І., Бобик І. О., Литвин В.В. Рівняння математичної фізики: навчальний посібник. Львів : «Новий світ-2000», 2010. 256 с.

28. Ярецька Н.О. Розв'язок контактної задачі для попередньо напружених циліндричного штампа та шару, що лежить без тертя на основі без початкових напружень. *Вісник Запорізького національного університету. Фізико-математичні науки*. Запоріжжя : Видавничий дім «Гельветика». 2021. №1 С. 90–100. https://doi.org/10.26661/2413-6549-2021-1-11

29. Ярецька Н. О. А. с. KNDS_CS_PZN. Комп'ютерна програма «Розрахунок компонентів напружено-деформованого стану для осесиметричної статичної задачі про тиск пружного циліндричного штампа на пружний шар з початковими (залишковими) напруженнями» № 54576; заявл. 05.05.2014; опубл. 01.09.2014, Бюл. № 34, 10 с.

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