EXPERT KNOWLEDGE ANALYSIS TECHNOLOGY WITH FUZZY PREFERENCE RELATION

Alyona Shved[1]

DOI: https://doi.org/10.30525/978-9934-26-241-8-7

Abstract. The adoption of managerial decisions occurring in complex social, economic, technical, organizational and other systems quite often occurs under a certain amount of conflicting factors that describe the most complex processes occurring in such systems. In this case, a decision-maker (expert) cannot guarantee an effective decision at the heuristic level, especially in a situation where it is difficult for an expert to express his/her preferences unambiguously, or the input data is not crisp, or is given with a certain degree of reliability, belonging to certain categories. To support the processes of preparation and synthesis of decisions under uncertainty (fuzziness), a well-developed apparatus of the fuzzy set theory and fuzzy relations is used, which makes it possible to correctly operate with various kinds of vague, fuzzy, uncertain concepts. *The purpose* of the paper is to consider one of the possible approaches to solving the problem of ordering multicriteria alternatives using fuzzy binary relations. The considered approach is based on the procedure for the synthesis of the resulting fuzzy binary relation as the intersection of the original fuzzy relations of a non-strict order. Based on the resulting fuzzy binary relation that satisfies the three conditions for the presence of a fuzzy order relation over it (the condition of reflexivity, antisymmetry and transitivity), the choice of the best alternative is made. Methodology of the study is based on general research methods of system analysis, the theory of decision-making, the theory of sets, the fuzzy set theory and fuzzy relations were used in the research process. The methods of the fuzzy set theory and fuzzy relations are used for fuzziness modeling; decision-making methods are applied when solving the problems of choosing the best alternative, ranking alternatives by significance. *Results* of the survey showed that the considered approach for multi-criteria problem solving, provided that the

[1] Doctor of Technical Sciences, Associate Professor at Department of Software Engineering, Petro Mohyla Black Sea National University, Ukraine

initial expert information is given in the form of fuzzy binary relations, becomes a theoretical basis for the synthesis of information technologies for intellectual resources (data and expert knowledge) managing with the aim of preparing information for making reasonable and effective decisions under incompleteness, uncertainty, and vagueness. *Practical implications* of the obtained results lie in the fact that the considered approach can be the basis of methods, algorithms and information technologies of intellectual decision-making support and its implementation as part of automated decision support systems in case when solving the task of ordering multi-criteria alternatives under inaccuracy, ambiguity and uncertainty. *Value/originality.* The proposed approach allows to model the uncertainty in expert judgments, through the presentation of inaccuracy (vagueness, fuzziness) in experts' assessments in the form of fuzzy binary relations. The algorithm of this approach is easily implemented computationally and can be useful for solving a number of practical problems.

1. Introduction

Currently, there are various approaches to solving one of the typical decision-making problems – determining the order (strict or non-strict) on the set of alternatives [1; 2, p. 11]. At the same time, the pairwise comparison method has become widespread, as a result of which a set of binary relations is obtained that reflects the relative importance of decisions.

To make decisions under uncertainty (inaccuracy, fuzziness), a well-developed apparatus of the fuzzy set theory is used, which makes it possible to correctly handle various kinds of vague, fuzzy concepts [3–6]. In recent decades, a new scientific direction has appeared, called the fuzzy relations theory.

The fuzziness of information is due to the presence in the description of decision-making problems of concepts and relations with non-strict boundaries, as well as statements with a multi-valued truth scale [7, p. 10]. An object may belong to a class described by some concept, relation, statement, or may not belong to it; at the same time, an object may belong to several classes simultaneously with varying degrees of membership. The concepts and relationships describing such classes are fuzzy.

Currently, there is a need to solve a fairly wide class of decision-making problems in which decision maker preferences, as well as the set of possible solutions, are fuzzy.

To describe fuzzy concepts in [5, p. 339], the concept of a fuzzy set theory is introduced, the membership function of which has the domain of definition in the range of [0, 1].

In general, the decision-making problem can be described by a tuple of the next form [7, p. 11]:

$$<A, Y, K, f, Ps; D, T>, \qquad (1)$$

where A is the set of alternatives; Y is the set of outcomes of alternatives; K is a vector-valued criterion for outcomes evaluation; f is the mapping of the set Y in the set of vector estimates; Ps is the decision maker's preference structure.

It is necessary to construct some decision rule (algorithm) D that allows performing the required action T on the set A. The rule (algorithm) D determines the principle of choosing elements from the set A in accordance with the required action T. In this case, each alternative $a \in A$ corresponds to a single outcome $y \in Y$, which is characterized by a vector estimate $K(x)$.

In a fuzzy environment, all elements of the problem (1) can be expressed in the form of fuzzy concepts and relations: alternatives, outcomes and dependencies between them, estimates of the probabilities of the occurrence of outcomes, criterial estimates of outcomes, decision maker preferences, decision rule [7, p. 11].

The solution to the problem of fuzzy multi-criteria choice is to find a fuzzy set of chosen solutions (best, optimal) that most fully satisfy the aspirations, interests and goals of the decision maker, with a given membership function $\mu(a)$.

The aim of the work is to consider possible approaches to solving the problem of ordering multicriteria alternatives using fuzzy binary relations.

2. Basic concepts and definitions of fuzzy relations

Fuzzy relations are an extension of ordinary relationships and correspondences to the case when the elements are in a given relationship or correspondence only with some degree of membership. This concept of the degree of membership fully corresponds to the concept of the degree of membership of an element in a fuzzy set.

When specifying an ordinary relation (correspondence), it is necessary to present all pairs of elements of a certain set that are definitely in this relation. In other words, the degree of membership of a pair of elements in a given relation is 1. A fuzzy relation differs from a regular relation in that different pairs of elements can be in a given relation with different degrees of membership. This definition of a fuzzy set is an extension of the notion of an ordinary relation [8, p. 391].

Let us formalize the concept of a fuzzy relation. Let there be two subsets $X \subset E_1$ and $Y \subset E_2$. Denote by P the Cartesian product of these subsets, $X \times Y$; by $M \in [0, 1]$ we denote the membership set of members of this product. A fuzzy subset P is called a fuzzy relation $\underset{\sim}{R}$. The fuzzy relation $\underset{\sim}{R}$ can be represented in the form [8, p. 391; 9, p. 35]:

$$x \in E_1,\ y \in E_2 : x \underset{\sim}{R}\, y, \tag{2}$$

Table 1 presents a fuzzy relation $\underset{\sim}{R}$ on the Cartesian product of universal sets $E_1 \times E_2$, where $X \subset E_1,\ X = \{x_1, x_2, x_3\}$ $Y \subset E_2,\ X = \{y_1, y_2, y_3\}$

Each cell of the Table 1 contains numbers that reflect the degree of belonging of the pairs (x_i, y_j), $i, j = 1, 2, 3$ to $\underset{\sim}{R}$ fuzzy relation.

Table 1

Fuzzy relation representation

$\underset{\sim}{R}$	y_1	y_2	y_3
x_1	0,7	0,1	0,3
x_2	0	1	0,5
x_3	0,4	0,2	0,3

The carrier of a fuzzy relation $\underset{\sim}{R}$ is the set S of ordered pairs (x, y) for which the membership function $\mu_{\underset{\sim}{R}}(x, y)$ has positive values [8, p. 394]:

$$S(\underset{\sim}{R}) = \{(x, y) / \mu_{\underset{\sim}{R}}(x, y) > 0\}. \tag{3}$$

For example, for the ordered pairs (x, y), presented in Table 1, the carrier of the fuzzy relation will be written as follows:

$$S(\underset{\sim}{R}) = \{(x_1, y_1), (x_1, y_2), (x_1, y_3), (x_2, y_2), (x_2, y_3), (x_3, y_1), (x_3, y_2), (x_3, y_3)\}.$$

Let there be two fuzzy relations $\underset{\sim}{R}$ and $\underset{\sim}{P}$ IF

$$\forall (x, y) \in E_1 \times E_2 : \mu_{\underset{\sim}{R}}(x, y) \le \mu_{\underset{\sim}{P}}(x, y),$$

THEN the relation $\underset{\sim}{P}$ contains the relation $\underset{\sim}{R}$ (or alternatively, the relation $\underset{\sim}{R}$ is contained in the relation $\underset{\sim}{P}$).

3. Operations over fuzzy relations

Let us consider a number of operations over fuzzy binary relations [8, p. 396; 10, p. 182; 11, p. 592].

3.1. Union of a fuzzy relations

Let $\underset{\sim}{R}$ and $\underset{\sim}{P}$ be two fuzzy relations. The values of the membership function of their union $\underset{\sim}{R} \cup \underset{\sim}{P}$ are determined by the expression:

$$\mu_{\underset{\sim}{R} \cup \underset{\sim}{P}}(x, y) = \max[\mu_{\underset{\sim}{R}}(x, y), \mu_{\underset{\sim}{P}}(x, y)]. \quad (4)$$

Tables 2 and 3 present the values of two fuzzy relations $\underset{\sim}{R}$ and $\underset{\sim}{P}$. It is necessary to determine their union $\underset{\sim}{R} \cup \underset{\sim}{P}$ (Table 4).

Table 2

Fuzzy relation $\underset{\sim}{R}$

$\underset{\sim}{R}$	y_1	y_2	y_3
x_1	0,7	0,3	0
x_2	0,2	1	0,6
x_3	1	0,8	0,1

$\cup$

Table 3

Fuzzy relation $\underset{\sim}{P}$

$\underset{\sim}{P}$	y_1	y_2	y_3
x_1	0,6	0,8	0,2
x_2	0,2	0,4	1
x_3	0,3	0,5	0,1

$\Rightarrow$

Table 4

Union of $\underset{\sim}{R}$ and $\underset{\sim}{P}$

$\underset{\sim}{R} \cup \underset{\sim}{P}$	y_1	y_2	y_3
x_1	0,7	0,8	0,2
x_2	0,2	1	1
x_3	1	0,8	0,1

3.2. Intersection of a fuzzy relations

Let $\underset{\sim}{R}$ and $\underset{\sim}{P}$ be two fuzzy relations. The value of the membership function of their intersection $\underset{\sim}{R} \cap \underset{\sim}{P}$ is defined as follows:

$$\mu_{\underset{\sim}{R} \cap \underset{\sim}{P}}(x, y) = \min[\mu_{\underset{\sim}{R}}(x, y), \mu_{\underset{\sim}{P}}(x, y)]. \quad (5)$$

Let us consider an example of fuzzy relation intersection (Tables 5, 6, 7).

Table 5

Fuzzy relation $\underset{\sim}{R}$

$\underset{\sim}{R}$	y_1	y_2	y_3
x_1	0,7	0,3	0
x_2	0,2	1	0,6
x_3	1	0,8	0,1

$\cap$

Table 6

Fuzzy relation $\underset{\sim}{P}$

$\underset{\sim}{P}$	y_1	y_2	y_3
x_1	0,6	0,8	0,2
x_2	0,2	0,4	1
x_3	0,3	0,5	0,1

$\Rightarrow$

Table 7

Intersection of $\underset{\sim}{R}$ and $\underset{\sim}{P}$

$\underset{\sim}{R} \cap \underset{\sim}{P}$	y_1	y_2	y_3
x_1	0,6	0,3	0
x_2	0,2	0,4	0,6
x_3	0,3	0,5	0,1

3.3. Complement of a fuzzy relation

The complement of a fuzzy relation $\underset{\sim}{R}$ is a fuzzy relation $\underset{\sim}{\bar{R}}$ with membership function values:

$$\forall (x, y) \in E_1 \times E_2 : \mu_{\underset{\sim}{\bar{R}}}(x, y) = 1 - \mu_{\underset{\sim}{R}}(x, y) . \tag{6}$$

Let us consider an example.

Table 8

Fuzzy relation $\underset{\sim}{R}$

$\underset{\sim}{R}$	y_1	y_2	y_3
x_1	0,7	0,3	0
x_2	0,2	1	0,6
x_3	1	0,8	0,1

$\Rightarrow$

Table 9

Complement of a fuzzy relation $\underset{\sim}{\bar{R}}$

$\underset{\sim}{\bar{R}}$	y_1	y_2	y_3
x_1	0,3	0,7	1
x_2	0,8	0	0,4
x_3	0	0,2	0,9

4. Properties of fuzzy binary relations

Let us consider a number of important properties of fuzzy binary relations [8, p. 415; 9, p. 37; 10, p. 182; 11, p. 592; 12, p. 16; 13, p. 156]. As before, we will assume that $E_1 \times E_2 = E$, that is, the relations are given on the same universal set.

4.1. Symmetry

A fuzzy binary relation $\underset{\sim}{R}$ is called symmetric IF

$$\forall (x, y) \in E \times E : (\mu_{\underset{\sim}{R}}(x, y) = \mu) \Rightarrow (\mu_{\underset{\sim}{R}}(y, x) = \mu) . \tag{7}$$

For a symmetric binary relation, the values of the membership function in the cells of the table, symmetrical with respect to its main diagonal, are the equal (Table 10).

Table 10

Symmetric fuzzy binary relation $\underset{\sim}{R}$

$\underset{\sim}{R}$	y_1	y_2	y_3	y_4
x_1	0,1	0,4	0,2	0
x_2	0,4	0,5	0,7	0,5
x_3	0,2	0,7	1	1
x_4	0	0,5	1	0

4.2. Reflexivity

A fuzzy binary relation $\underset{\sim}{R}$ is called reflexive IF

$$\forall (x, y) \in E \times E : \mu_{\underset{\sim}{\bar{R}}}(x, x) = 1 . \quad (8)$$

For a reflexive binary relation, the values of the membership function in the cells on the main diagonal (Table 11) should contain only 1.

Table 11

Reflexive fuzzy binary relation $\underset{\sim}{R}$

$\underset{\sim}{R}$	y_1	y_2	y_3	y_4
x_1	1	0,4	0,3	0
x_2	0	1	0,7	0,7
x_3	0,4	0,3	1	0
x_4	0	0,8	0,5	1

4.3. Transitivity

A fuzzy binary relation is called transitive (Table 12) IF

$$\forall (x, y), (y, z), (z, x) \in E \times E : \mu_{\underset{\sim}{R}}(x, z) \geq \max[\min(\mu_{\underset{\sim}{R}}(x, y), \mu_{\underset{\sim}{R}}(y, z))] . \quad (9)$$

Table 12

Transitive fuzzy binary relation $\underset{\sim}{R}$

$\underset{\sim}{R}$	y_1	y_2	y_3	y_4
x_1	0,2	1	0,4	0,4
x_2	0	0,6	0,3	0
x_3	0	1	0,3	0
x_4	0,1	1	1	0,1

4.4. Antisymmetry

A fuzzy binary relation is called antisymmetric (Table 13) IF

$$\forall (x, y) \in E \times E, x \neq y : \mu_{\underset{\sim}{R}}(x, y) \neq \mu_{\underset{\sim}{R}}(y, x) \vee \mu_{\underset{\sim}{R}}(x, y) = \mu_{\underset{\sim}{R}}(y, x) = 0 . \quad (10)$$

Table 13

Antisymmetric fuzzy binary relation $\underset{\sim}{R}$

$\underset{\sim}{R}$	y_1	y_2	y_3
x_1	1	0	0,1
x_2	0	0,4	0,2
x_3	0,8	0,4	0,2

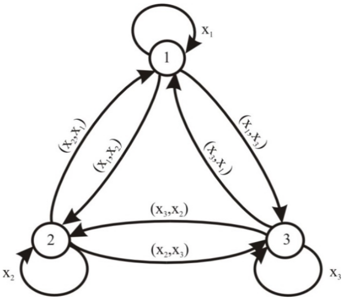
5. Checking the transitivity of fuzzy binary relations

One of the typical decision-making problems is to determine the order (strict or non-strict) on a set of alternatives, which means estimating the relative importance of each of them. In this case, an important condition for solving such a problem is to check the resulting orderings for transitivity. Examples of transitive relations are the relations "x_i is greater than x_j" or "x_i is less than x_j". A relation R over set X is a relation of strict order if it is transitive and antisymmetric (i.e., for any elements x_i, $x_j \in X$, it cannot be that $x_i R x_j$ and $x_j R x_i$). Using this relation, the elements of the set X can be ordered according to some given attribute. A relation R over set X is a relation of non-strict order if it is transitive, antisymmetric and reflexive (for any $x_i \in X$, $x_i R x_i$).

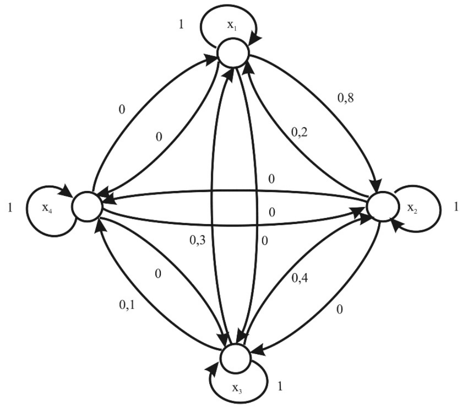
Let us check $\underset{\sim}{R}_0$ on transitivity according to the scheme proposed in [8, p. 418; 13, p. 160]. First of all, it is necessary to set $\underset{\sim}{R}_0$ in the form of a table that displays in numerical form the degree of membership of pairs (x_i, x_j) in a given fuzzy binary relation, and build a fuzzy graph using these values (Figure 1).

Further, for each vertex of the graph, it is necessary to compare the value of the membership function $\mu_{\underset{\sim}{R}_0}$ for the arc leaving this vertex and entering it with the values of the same function for each set of arcs along which it is possible to exit and reach the considered vertex again. Similarly, the value $\mu_{\underset{\sim}{R}_0}$ for each arc connecting two vertices must be compared with the values of the same function for all possible sets of arcs connecting a pair of vertices in a fuzzy graph. For this, various compositions of fuzzy relations are used according to the maximin rule (9).

Let some fuzzy relation be given, for example, in the form of a table that displays numerically the degree of belonging (the membership degree) of pairs (x_i, x_j) to this fuzzy relation (Table 14).



a)



b)

Figure 1. A fuzzy graph displaying a given fuzzy relation *R*:

a) for case $X = \{x_i | i = \overline{1,3}\}$; b) for case $X = \{x_i | i = \overline{1,4}\}$

Comparison of the values of the membership function is carried out according to the condition (9). Previously, based on the Table 14, it can be established that given $\underset{\sim}{R}$ is reflexive (only ones are located on the main diagonal of the matrix) and antisymmetric, which is easily verified by comparing the values $\mu_{\underset{\sim}{R}}(x_i, x_j)$, $\mu_{\underset{\sim}{R}}(x_j, x_i)$, i, j = 1, 2, 3, 4.

Table 14

Fuzzy relation $\underset{\sim}{R}$

$\underset{\sim}{R}$	x_1	x_2	x_3	x_4
x_1	1	0,8	0	0
x_2	0,2	1	0	0
x_3	0,3	0,4	1	0,1
x_4	0	0	0	1

Let's perform the necessary calculations to determine the transitivity or non-transitivity of the given $\underset{\sim}{R}$:

1) For the arc (x_1, x_1):

$$\min[\mu_{\tilde{R}}(x_1,x_1),\mu_{\tilde{R}}(x_1,x_1)] = \min(1;1) = 1;$$
$$\min[\mu_{\tilde{R}}(x_1,x_2),\mu_{\tilde{R}}(x_2,x_1)] = \min(0,8;0,2) = 0,2;$$
$$\min[\mu_{\tilde{R}}(x_1,x_3),\mu_{\tilde{R}}(x_3,x_1)] = \min(0;0,3) = 0;$$
$$\min[\mu_{\tilde{R}}(x_1,x_4),\mu_{\tilde{R}}(x_4,x_1)] = \min(0;0) = 0;$$
$$\max(1;0,2;0;0) = 1;$$
$$\mu_{\tilde{R}}(x_1,x_1) = 1 = 1.$$

2) For the arc (x_1, x_2):

$$\min[\mu_{\tilde{R}}(x_1,x_1),\mu_{\tilde{R}}(x_1,x_2)] = \min(1;\ 0,8) = 0,8;$$
$$\min[\mu_{\tilde{R}}(x_1,x_2),\mu_{\tilde{R}}(x_2,x_2)] = \min(0,8;1) = 0,8;$$
$$\min[\mu_{\tilde{R}}(x_1,x_3),\mu_{\tilde{R}}(x_3,x_2)] = \min(0;0,4) = 0;$$
$$\min[\mu_{\tilde{R}}(x_1,x_4),\mu_{\tilde{R}}(x_4,x_2)] = \min(0;0) = 0;$$
$$\max(0,8;0,8;0;0) = 0,8;$$
$$\mu_{\tilde{R}}(x_1,x_2) = 0,8 = 0,8.$$

3) For the arc (x_1, x_3):

$$\min[\mu_{\tilde{R}}(x_1,x_1),\mu_{\tilde{R}}(x_1,x_3)] = \min(1;\ 0) = 0;$$
$$\min[\mu_{\tilde{R}}(x_1,x_2),\mu_{\tilde{R}}(x_2,x_3)] = \min(0,8;0) = 0;$$
$$\min[\mu_{\tilde{R}}(x_1,x_3),\mu_{\tilde{R}}(x_3,x_3)] = \min(0;1) = 0;$$
$$\min[\mu_{\tilde{R}}(x_1,x_4),\mu_{\tilde{R}}(x_4,x_3)] = \min(0;0,1) = 0;$$
$$\max(0;0;0;0) = 0;$$
$$\mu_{\tilde{R}}(x_1,x_3) = 0 = 0.$$

4) For the arc (x_1, x_4):

$$\min[\mu_{\tilde{R}}(x_1,x_1),\mu_{\tilde{R}}(x_1,x_4)] = \min(1;\ 0) = 0;$$
$$\min[\mu_{\tilde{R}}(x_1,x_2),\mu_{\tilde{R}}(x_2,x_4)] = \min(0,8;0) = 0;$$
$$\min[\mu_{\tilde{R}}(x_1,x_3),\mu_{\tilde{R}}(x_3,x_4)] = \min(0;0,1) = 0;$$
$$\min[\mu_{\tilde{R}}(x_1,x_4),\mu_{\tilde{R}}(x_4,x_4)] = \min(0;1) = 0;$$
$$\max(0;0;0;0) = 0;$$
$$\mu_{\tilde{R}}(x_1,x_4) = 0 = 0.$$

Similar actions are performed for other graph arcs (Figure 1, *b*).

5) For the arc (x_2, x_1).

$\max(0,2;0,2;0;0)=0,2;$

$\mu_{\tilde{R}}(x_2,x_1)=0,2=0,2.$

6) For the arc (x_2, x_2).

$\max(0,2;1;0;0)=1;$

$\mu_{\tilde{R}}(x_2,x_2)=1=1.$

7) For the arc (x_2, x_3).

$\max(0;0;0;0)=0;$

$\mu_{\tilde{R}}(x_2,x_3)=0=0.$

8) For the arc (x_2, x_4).

$\max(0;0;0;0)=0;$

$\mu_{\tilde{R}}(x_2,x_4)=0=0.$

9) For the arc (x_3, x_1).

$\max(0,3;0,2;0,3;0)=0,3;$

$\mu_{\tilde{R}}(x_3,x_1)=0,3.$

10) For the arc (x_3, x_2).

$\max(0,3;0,4;0,4;0)=0,3;$

$\mu_{\tilde{R}}(x_3,x_2)=0,4=0,4.$

11) For the arc (x_3, x_3).

$\max(0;0;1;0)=1;$

$\mu_{\tilde{R}}(x_3,x_3)=1=1.$

12) For the arc (x_3, x_4).

$\max(0;0;0,1;0,1)=0,1;$

$\mu_{\tilde{R}}(x_3,x_4)=0,1=0,1.$

13) Semicircular arc (x_4, x_1).

$\max(0;0;0;0)=0;$

$\mu_{\tilde{R}}(x_4,x_1)=0=0.$

14) For the arc (x_4, x_2).

$\max(0;0;0;0)=0;$

$\mu_{\underset{\sim}{R}}(x_4, x_2) = 0 = 0.$

15) For the arc (x_4, x_3).

$\max(0;0;0;0) = 0;$

$\mu_{\underset{\sim}{R}}(x_4, x_3) = 0 = 0.$

16) For the arc (x_4, x_4).

$\max(0;0;0;1) = 1;$

$\mu_{\underset{\sim}{R}}(x_4, x_4) = 1 = 1.$

Based on the above calculations, it can be concluded that condition (9) is satisfied for all analyzed arcs, i.e. fuzzy binary relation $\underset{\sim}{R}$ is transitive.

6. Decision making with a fuzzy preference relation

Let us consider the problem of ordering multicriteria alternatives using fuzzy binary relations. Let a set of m alternative solutions $X = \{x_i | i = \overline{1,m}\}$ be given, which is evaluated using n criteria $K = \{k_j | j = \overline{1,n}\}$. It is assumed that a decision maker (DM) or an expert can compare all possible pairs of decisions $(x_i, x_j) \in X$ by preference.

The results of such a comparison for each of the criteria can be reflected in the form of fuzzy binary relation $\underset{\sim}{R}_j$, $j = 1,..,n$:

$$\begin{array}{cc} & \begin{array}{cccc} \underset{\sim}{R}_j & x_1 & \ldots & x_m \end{array} \\ \mu_k = & \begin{array}{c|ccc|} x_1 & \mu_{11} & \ldots & \mu_{1m} \\ \vdots & \ldots & \ldots & \ldots \\ x_m & \mu_{m1} & \ldots & \mu_{mm} \end{array} \end{array} \quad (11)$$

A fuzzy relation R over X set is a fuzzy subset of the Cartesian product $X \times X$ characterized by a preference function $\mu_R : X \times X \to [0, 1]$. The value $\mu_R(x, y) \in [0,1]$ is understood as the degree of fulfillment of the relation $x\ R\ y$ (the degree of confidence that the object x is in a given relation with the object y), thus the membership function of the fuzzy relation R for each pair $(x, y) \in X$ determines the certainty $\mu_R(x, y)$ that the object (alternative) x is not worse than the object y ($x \succeq y$) [9, p. 35; 13, p. 156].

Binary relations are used to describe pairwise relationships of a different nature between objects of an arbitrary nature [9, p. 34].

A fuzzy non-strict preference relation over X set is any reflexive fuzzy relation defined on this set (in this case, it is assumed that any object is reliably not worse than itself) [8, p. 436; 9, p. 40].

A strong preference fuzzy relation $R^S = R \setminus R^{-1}$ (where R^{-1} is the "inverse" relation; the relation matrix R^{-1} is obtained by transposing the relation matrix R) over X set is any antireflexive and antisymmetric fuzzy relation [9, p. 40].

The membership function of a strict order fuzzy relation according to the j-th criterion over the X set is defined as

$$\mu_{\tilde{R}_j^S}(x,y) = \max[\mu_{\tilde{R}_j}(x,y) - \mu_{\tilde{R}_j}(y,x); 0] \tag{12}$$

In this case, we can say that object x dominates object y ($x \succ y$) with certainty $\mu_R(x, y)$.

A fuzzy indifference (equivalence) relation over X set is any reflexive and symmetric fuzzy relation such that $R^E = ((X \times X)\backslash(R \cup R^{-1})) \cup (R \cap R^{-1})$ [9, p. 40].

The membership function of a fuzzy indifference relation over X set is defined as

$$\mu_{\tilde{R}_j^E}(x,y) = \max[\min[1-\mu_{\tilde{R}_j}(x,y); 1-\mu_{\tilde{R}_j}(y,x)]; \min[\mu_{\tilde{R}_j}(x,y); \mu_{\tilde{R}_j}(y,x)]]. \tag{13}$$

An important property of the preference relation is its linearity (completeness):

$$\max[\mu_{\tilde{R}_j}(x,y), \mu_{\tilde{R}_j}(y,x)] = 1. \tag{14}$$

Property (14) guarantees that the decision maker has enough information to compare any objects (alternatives).

To solve the problem of choosing rational solutions, it is necessary to compare alternative solutions across the entire set of criteria. Such a comparison can, in principle, be performed using an intersection procedure of the form $Q_1 = \bigcap_{j=1}^{n} \underset{\sim}{R}_j$ [8, p. 437; 14, p. 63; 15, p. 32; 16]. The result is the next fuzzy binary preference relation $\underset{\sim}{R}_0$:

$$\mu_{R_0 = \underset{\sim}{R}_1 \cap \ldots \cap \underset{\sim}{R}_n}(x_i, x_j) = \min\left[\mu_{\underset{\sim}{R}_1}(x_i,x_j), \mu_{\underset{\sim}{R}_2}(x_i,x_j), \ldots, \mu_{\underset{\sim}{R}_n}(x_i,x_j)\right] \tag{15}$$

If one of the typical decision-making problems related to determining the order (strict or non-strict) on a set of alternatives is being solved, then the resulting fuzzy binary relation must be checked for reflexivity, antisymmetry, and transitivity.

The fulfillment of all above conditions (7), (8) and (9) allows to assert that on the set of ordered pairs of fuzzy binary relations (x_i, x_j) there must be a fuzzy order relation, which can be obtained by forming a certain number of preference schemes (according to the number considered alternative solutions) and checking them for transitivity [8, p. 435].

7. The selection of technological process of cutting and welding technologies based on fuzzy binary relations

Let's give a numerical example. Let $X = \{x_i | i = \overline{1,m}\}$, m=3 be considered a set of alternative solutions (options for the technological process of cutting and welding technologies), which must be evaluated according to $K = \{k_j | j = \overline{1,n}\}$, n=3 criteria. As criteria for the selection of welding and cutting technologies, the following can be considered: technical capabilities, operational reliability, ease of maintenance, types and amount of energy required for the operation of the device, equipment maintenance expenses, welding quality, etc.

It is necessary to determine the optimal, from the point of view of the considered criteria and the obtained technical and economic indicators, the variant of the technological process of cutting and welding x_i.

Representatives of staff units (chief designer's department, chief technologist's department, planning and production department, material and technical supply department, etc.), as well as representatives of line units (manufacturing workshops, divisions, etc.) can act as an expert or decision maker (DM).

The DM or experts expressed their fuzzy preferences over set of alternatives for each of the criteria with different values of the membership function $\mu \in [0, 1]$ in the form of the following matrices:

$$\begin{array}{c} \underset{\sim}{R}_1 \\ \mu_{k_1} = \end{array}\begin{array}{c|ccc|} & x_1 & x_2 & x_3 \\ \hline x_1 & 1 & 0,9 & 0,1 \\ x_2 & 0,3 & 1 & 0,4 \\ x_3 & 0,6 & 0,7 & 1 \end{array} \quad \begin{array}{c} \underset{\sim}{R}_2 \\ \mu_{k_2} = \end{array}\begin{array}{c|ccc|} & x_1 & x_2 & x_3 \\ \hline x_1 & 1 & 0,8 & 0,3 \\ x_2 & 0,6 & 1 & 0,1 \\ x_3 & 0,4 & 0,5 & 1 \end{array} \quad \begin{array}{c} \underset{\sim}{R}_3 \\ \mu_{k_3} = \end{array}\begin{array}{c|ccc|} & x_1 & x_2 & x_3 \\ \hline x_1 & 1 & 0,8 & 0 \\ x_2 & 0,2 & 1 & 0 \\ x_3 & 0,3 & 0,4 & 1 \end{array}$$

Let us perform the operation $\underset{\sim}{R}_1 \cap \underset{\sim}{R}_2 \cap \underset{\sim}{R}_3$ taking into account the condition (15), as a result of which we obtain the resulting fuzzy binary relation $\underset{\sim}{R}_0$, for which a fuzzy graph will be constructed (Figure 1, *a*):

$$\begin{array}{c|ccc|}
\underset{\sim 0}{R} & x_1 & x_2 & x_3 \\
x_1 & 1 & 0,8 & 0 \\
\mu = x_2 & 0,2 & 1 & 0 \\
x_3 & 0,3 & 0,4 & 1
\end{array}$$

Let us compare the values of μ both for the vertices of the graph and for their possible pairs.

1. For vertex 1:

$\min[\mu(x_1,x_1),\mu(x_1,x_1)] = \min(1;1) = 1;$

$\min[\ \mu(x_1,x_2),\mu(x_2,x_1)\ ] = \min(0,8;0,2) = 0,2;$

$\min[\ \mu(x_1,x_3),\mu(x_3,x_1)\ ] = \min(0;0,3) = 0;$

$\max(1;0,2;0) = 1;$

$\mu(x_1,x_1) = 1 = 1.$

2. For vertex 2:

$\min[\ \mu(x_2,x_2),\mu(x_2,x_2)\ ] = \min(1;1) = 1;$

$\min[\ \mu(x_2,x_1),\mu(x_1,x_2)\ ] = \min(0,2;0,8) = 0,2;$

$\min[\ \mu(x_2,x_3),\mu(x_3,x_2)\ ] = \min(0;0,4) = 0;$

$\max(1;0,2;0) = 1;$

$\mu(x_2,x_2) = 1 = 1.$

3. For vertex 3:

$\min[\ \mu(x_3,x_3),\mu(x_3,x_3)\ ] = \min(1;1) = 1;$

$\min[\mu(x_3,x_1),\mu(x_1,x_3)\ ] = \min(0,3;0) = 0;$

$\min[\mu(x_3,x_2),\mu(x_2,x_3)\ ] = \min(0,4;0) = 0;$

$\max(1;0;0) = 1;$

$\mu(x_3,x_3) = 0 = 0.$

4. For vertices 1 and 2:

$\max\{\min[\ \mu(x_1,x_2),\mu(x_2,x_1)\ ]\} = \max\{\min(0,8;0,2)\} = 0,2;$

$\mu(x_1,x_2) = 0,8 > 0,2.$

5. For vertices 1 and 3:

$\max\{\min[\ \mu(x_1,x_3),\mu(x_3,x_1)\ ]\} = \max\{\min(0;0,3)\} = 0;$

$\mu(x_1,x_3) = 0 = 0.$

6. For vertices 2 and 3:

$\max\{\min[\mu(x_2,x_3),\mu(x_3,x_2)]\} = \max\{\min(0;0,4)\} = 0;$

$\mu(x_2,x_3) = 0 = 0.$

Thus, the investigated resulting fuzzy binary relation $\underset{\sim}{R}_0$ satisfies all three conditions for the presence of a fuzzy order (non-strict) relation.

Let's search for it, having previously formed the following 6 possible (for $n = 3$) preference schemes:

$$x_1 \succ x_2 \succ x_3; \quad x_2 \succ x_3 \succ x_1; \quad x_3 \succ x_1 \succ x_2;$$
$$x_2 \succ x_1 \succ x_3; \quad x_1 \succ x_3 \succ x_2; \quad x_3 \succ x_2 \succ x_1. \tag{16}$$

Let us sequentially check schemes (16) on transitivity:

1. $x_1 \succ x_2 \succ x_3$:

IF $\{(x_1 \xrightarrow{\mu_{12}=0,8} x_2) \succ (x_2 \xrightarrow{\mu_{21}=0,2} x_1)\} \wedge$

$\wedge\{(x_2 \xrightarrow{\mu_{23}=0} x_3) \succ (x_3 \xrightarrow{\mu_{32}=0,4} x_2)\}$,

THEN $\{(x_1 \xrightarrow{\mu_{13}=0} x_3) \succ (x_3 \xrightarrow{\mu_{31}=0,3} x_1)\}$;

2. $x_2 \succ x_3 \succ x_1$:

IF $\{(x_2 \xrightarrow{\mu_{23}=0} x_3) \succ (x_3 \xrightarrow{\mu_{32}=0,4} x_2)\} \wedge$

$\wedge\{(x_3 \xrightarrow{\mu_{31}=0,3} x_1) \succ (x_1 \xrightarrow{\mu_{13}=0} x_3)\}$,

THEN $\{(x_2 \xrightarrow{\mu_{21}=0,2} x_1) \succ (x_1 \xrightarrow{\mu_{12}=0,8} x_2)\}$;

3. $x_3 \succ x_1 \succ x_2$:

IF $\{(x_3 \xrightarrow{\mu_{31}=0,3} x_1) \succ (x_1 \xrightarrow{\mu_{13}=0} x_3)\} \wedge$

$\wedge\{(x_1 \xrightarrow{\mu_{12}=0,8} x_2) \succ (x_2 \xrightarrow{\mu_{21}=0,2} x_1)\}$,

THEN $\{(x_3 \xrightarrow{\mu_{32}=0,4} x_2) \succ (x_2 \xrightarrow{\mu_{23}=0} x_3)\}$;

4. $x_2 \succ x_1 \succ x_3$:

IF $\{(x_2 \xrightarrow{\mu_{21}=0,2} x_1) \succ (x_1 \xrightarrow{\mu_{12}=0,8} x_2)\} \wedge$

$\wedge\{(x_1 \xrightarrow{\mu_{13}=0} x_3) \succ (x_3 \xrightarrow{\mu_{31}=0,3} x_1)\}$,

THEN $\{(x_2 \xrightarrow{\mu_{23}=0} x_3) \succ (x_3 \xrightarrow{\mu_{32}=0,4} x_2)\}$;

5. $x_1 \succ x_3 \succ x_2$:

IF $\{(x_1 \xrightarrow{\mu_{13}=0} x_3) \succ (x_3 \xrightarrow{\mu_{31}=0,3} x_1)\} \wedge$

$$\wedge\{(x_3 \xrightarrow{\mu_{32}=0,4} x_2) \succ (x_2 \xrightarrow{\mu_{23}=0} x_3)\},$$

THEN $\{(x_1 \xrightarrow{\mu_{12}=0,4} x_2) \succ (x_2 \xrightarrow{\mu_{21}=0,3} x_1)\};$

6. $x_3 \succ x_2 \succ x_1$:

IF $\{(x_3 \xrightarrow{\mu_{32}=0,4} x_2) \succ (x_2 \xrightarrow{\mu_{23}=0} x_3)\} \wedge \{(x_2 \xrightarrow{\mu_{21}=0,2} x_1) \succ (x_1 \xrightarrow{\mu_{12}=0,8} x_2)\},$

THEN $\{(x_3 \xrightarrow{\mu_{31}=0,3} x_1) \succ (x_1 \xrightarrow{\mu_{13}=0} x_3)\};$

The third preference scheme $x_3 \succ x_1 \succ x_2$ is transitive and hence the alternative x_3 is the best.

8. Conclusions

Currently, there are various approaches to solving one of the typical decision-making problems – determining the order (strict or non-strict) on a set of alternatives. The most widely used is the method of pairwise comparison of alternatives, the results of which are often presented in the form of a binary relation, which make it possible to display the relative degree of importance of the analyzed objects. It is believed that it is much easier to make a qualitative comparison of two objects than to express one's preferences in a point or rank scale. In this case, in real practice, situations may arise in which it is quite difficult for an expert to unambiguously determine the belonging of objects to some binary relation. To make decisions under such kind of uncertainty (inaccuracy, fuzziness), a well-developed apparatus of the fuzzy set theory is used, which makes it possible to correctly handle various kinds of vague and fuzzy concepts. Fuzzy relations are an extension of ordinary relations in case when elements are in a given relation only with some degree of membership. Which, in turn, allows modeling uncertainty (fuzziness) in the expert's judgments.

The basic concepts of fuzzy sets and fuzzy binary relations has been considered in the paper. Their properties are investigated. The main operations on fuzzy binary relations are considered: checking binary relations for transitivity, reflexivity, symmetry, which underlie the procedure for determining the order (strict or non-strict) over the set of initial objects (alternatives).

The paper considers the technology of analysis of expert assessments formed as a result of pairwise comparison of objects of expertise and presented in the form of a fuzzy preference relation, which is based on the procedures for checking them on reflexivity, antisymmetry and transitivity.

References:

1. Larichev O. I. (2003) Teoriya i metody prinyatiya resheniy [Decision theory and methods]. Moscow: Logos. (in Russian)

2. Mirkin B. G (1974) Problema gruppovogo vybora [Group choice problem]. Moscow: Nauka. (in Russian)

3. Borisov A. N., Krumberg O. A., Fedorov I. P. (1990) Prinjatie reshenij na osnove nechetkih modelej. Primery ispol'zovanija [Decision making based on fuzzy models. Examples]. Riga: Zinatne. (in Russian)

4. Zade L. (1979) Lingvisticheskaja peremennaja i teorija nechetkih mnozhestv [Linguistic variable and fuzzy set theory]. Moscow: Mir. (in Russian)

5. Zadeh L. A. (1965) Fuzzy sets. *Information and Control*, vol. 8(3), pp. 338–353. DOI: https://doi.org/10.1016/S0019-9958(65)90241-X

6. Kofman A. (1982) Vvedenie v teoriju nechetkih mnozhestv [Introduction to Fuzzy Set Theory]. Moscow: Radio i svjaz'. (in Russian)

7. Borisov A. N., Alekseev A. B., Merkur'ev G. V. (1989) Obrabotka nechetkoj informacii v sistemah prinjatija reshenij [Processing of fuzzy information in decision-making systems]. Moscow: Radio i svjaz'. (in Russian)

8. Uzga-Rebrovs O. (2010) Nenoteiktiby parvaldisana. Rezekne: RA Izdevnieciba, vol. 3.

9. Lotov A. V., Pospelova I. I. (2008) Mnogokriterial'nye zadachi prinjatija reshenij [Multicriteria decision making problems]. Moscow: MAKS Press. (in Russian)

10. Bělohlávek R. (2002) Binary fuzzy relations. In: *Fuzzy Relational Systems. International Federation for Systems Research International Series on Systems Science and Engineering*, vol. 20. Boston: Springer, pp. 181–213. DOI: https://doi.org/10.1007/978-1-4615-0633-1_4

11. Rajalakshmi R. (2021) Fuzzy binary relations and its specials types. *International Research Journal of Modernization in Engineering Technology and Science*, vol. 3(1), pp. 590–601.

12. Smarandache F., Kandasamy W. (2014) Fuzzy relations maps and neutrosophic relational maps. doi: 10.6084/M9.FIGSHARE.1015555. Available at: https://arxiv.org/ftp/math/papers/0406/0406622.pdf (accessed September 28, 2022).

13. Orlovsky S. A. (1978) Decision-making with a fuzzy preference relation. *Fuzzy Sets and Systems*, vol. 1(3), pp. 155–167. DOI: https://doi.org/10.1016/0165-0114(78)90001-5

14. Bljumin C. L., Shujkova I. A. (2001) Modeli i metody prinjatija reshenij v uslovijah neopredelennosti [Models and methods of decision making under uncertainty]. Lipeck: LJeGI. (in Russian)

15. Zhukovin V. E. (1988) Nechetkie mnogokriterial'nye modeli prinjatija reshenij [Fuzzy multicriteria decision making models]. Tbilisi: Mecniereba. (in Russian)

16. Orlovskij S. A. (1981) Problemy prinjatija reshenij pri nechetkoj ishodnoj informacii [Decision-Making Problems with Fuzzy Initial Information]. Moscow: Nauka. (in Russian)