THE METHOD OF INTEGRATING SYSTEMS OF HIGH-ORDER EQUILIBRIUM EQUATIONS OF THE MATHEMATICAL THEORY OF THICK PLATES UNDER INTERMITTENT LOADS (PART 2)

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Abstract. The subject of the study is the development of an analytical method for solving boundary value problems of a variant of the mathematical theory of transversally isotropic plates of arbitrary constant thickness, which boil down to the integration of systems of inhomogeneous high-order differential equations of equilibrium. According to the developed version of the theory, all components of the stress-strain state and boundary conditions are considered functions of three coordinates. The indicated functions of the three variables are expanded into infinite mathematical series by Legendre polynomials from the transverse coordinate. The boundary conditions on the front faces of the plates are fulfilled exactly. The boundary conditions on the lateral surfaces are fulfilled according to the appropriate approximation, which is determined by a certain number of terms in partial sums of mathematical series. This makes it possible to effectively determine all components of the stress-strain state with any high accuracy. It is also important to note that the developed version of the mathematical theory takes into account vortex and potential edge effects with high accuracy. The problem here lies in the high-order systems of differential equilibrium equations with partial derivatives. This complicates their solution. Moreover, the order of systems increases with an increase in the number of members in the partial mathematical sums of the development of components of the stress-strain state into infinite mathematical series. Therefore, the solution of this problem is connected with the application of a new methodology for finding partial and general solutions. The

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methodology consists in the fact that the initial systems of high-order differential equations of equilibrium are reduced by various mathematical transformations to convenient homogeneous and heterogeneous systems of high-order differential equations. These equations, in turn, are reduced to homogeneous and inhomogeneous differential equations of the second order by the developed operator method. The general and partial solutions of the initial equilibrium systems are expressed through the general and partial solutions of the second-order differential equations. The skew-symmetric transverse load of the plates relative to the median plane is considered. The goal is to obtain analytical solutions of boundary value problems for plates of arbitrary constant thickness. General solutions in special functions for components of the stress-strain state (SSS) from the annular transverse load of circular and annular plates under axisymmetric deformation are obtained. Analytical solutions of the boundary value problems for the specified plates under the action of axisymmetric loads for various static and kinematic boundary conditions on the lateral surfaces were obtained. An analysis of the obtained results was carried out.

The proposed methodology, the method of solving systems of high-order differential equations of equilibrium, the method of obtaining general and partial solutions can also be applied in classical and refined theories, including theories of the Tymoshenko-Reissner type.

1. Introduction

Plates and shells are used in various objects of energy, mechanical engineering, construction and other branches of modern industry. Ensuring the reliable operation of such structures requires the use of high-precision theories for their calculation and adequate mathematical methods for solving the corresponding limit problems, which would take into account all components of the SSS state as a function of three variables, and edge effects.

Calculations based on classical theories of non-thin plates and shells, in cases with non-smooth, local and concentrated loads, in the presence of significant anisotropy and in other cases, give results that may differ significantly from the exact ones according to the three-dimensional theory of elasticity [1, p. 3; 2, p. 51; 3, p. 69; 4, p. 64; 5, p. 99; 6, p. 382; 7, p. 147; 8, p. 84; 9, p. 569; 10, p. 84; 11, p. 63; 12, p. 67; 13, p.127; 14, p. 54].

Refined theories, which are based on various hypotheses, as well as theories of the Tymoshenko-Reissner type [15, p. 242; 16; 17, p. 486; 18, p. 239; 19; 20, p. 675; 21, p. 288; 22, p. 423; 23, p. 238; 24, p. 663; 25, p. 993; 26, p. 195; 27; 28, p. 184; 29, p. 504; 30, p. 744; 31], are mainly used today in the works of domestic and especially foreign authors. In each case, these theories require a justified use and establishing the frameworks of their suitability for solving boundary problems. Finding the SSS of plates and shells according to these theories cannot be performed with arbitrarily high accuracy, since their accuracy is determined by the accepted hypotheses.

Existing variants of mathematical theory (MT) [31; 32, p. 238; 33; 34, p. 83; 35; 36, p. 77; 37; 38, p. 3; 39, p. 335; 40, p. 21; 41, p. 191; 42, p. 154; 43, p. 60; 44, p. 21; 45, p. 496; 46, p. 51; 47, p. 741; 48, p. 78; 49, p. 221; 50, p. 27; 51, p. 49], which are based on the development of the SSS components in endless mathematical series, require the possibility of analytical solution of the obtained systems of differential equilibrium equations and obtaining numerical results. The accuracy of variants of MT depends on the methodology of obtaining basic equations and the accuracy of satisfaction of marginal conditions. Solving marginal problems for plates and shells in a three-dimensional setting [52; 53; 54, p. 3; 55, p. 49; 56, p. 22] is associated with sufficient mathematical difficulties.

The relevance of the problem is to build a variant of MT of plates of arbitrary constant thickness, which would describe with high accuracy of their SSS with arbitrary static loads, and the development of effective analytical methods of integration of the obtained systems of differential equations of high orders in boundary problems. The novelty of the work is to solve this scientific problem. In the works of the author, the problem of constructing a new variant of the MT of transversally isotropic plates of arbitrary constant thickness is solved.

This work uses the operator method, according to which the inhomogeneous differential equations of high order with partial derivatives are reduced to the inhomogeneous differential equations of the second order. A new methodology of integration of systems of inhomogeneous differential equations of equilibrium of transverse-isotropic plates of high orders has been developed.

The purpose of the work is: obtaining general solutions of systems of inhomogeneous differential equations of equilibrium of high orders of the MT variant in axisymmetric boundary value problems for circular plates of arbitrary constant thickness with jump-like loads; obtaining analytical solutions of boundary value problems for ring and circular plates under different boundary conditions on the side surface.

2. Statement of the problem

The developed variant of the mathematical theory [9, p. 569; 10, p. 84; 11, p. 63; 12, p. 67; 13, p. 127; 14, p. 54; 44, p. 21; 45, p. 968] makes it possible to efficiently and with high accuracy determine the SSS of transversally isotropic plates of arbitrary constant thickness and take into account edge effects. According to this variant, boundary value problems for plates can be solved with any predefined accuracy. Moreover, unlike the exact solution based on the three-dimensional equations of the theory of elasticity, the constructed variant of the mathematical theory provides a real possibility of analytically solving boundary value problems for any boundary conditions on the lateral surface.

In addition, obtaining an analytical solution based on the specified variant of MT is easier than from the standpoint of the three-dimensional theory of elasticity. But the mathematical complexity is still great. Moreover, it increases with an increase in the number of terms in partial sums of mathematical series for SSS components. The effectiveness and high accuracy of the developed version of the mathematical theory is shown, in particular, in the author's works.

Thus, for the built high-precision variant of the MT, a problem arises – a mathematical problem, which consists in the need to develop effective analytical methods for solving systems of high-order differential equations of equilibrium, to which boundary value problems.

2.1. The essence of the variant of the mathematical theory. Methodology for building the MT variant. The essence of the variant of MT plates of arbitrary constant thickness and the methodology of its construction is described in [49, p. 221], and therefore we will not dwell on it in detail. However, we will give some important relations and equations later in paragraphs 2.2, 2.3, some of which are missing in [49, p. 223**]**.

2.2. Reissner's variational equation and boundary conditions on front faces. The three-dimensional problem of the theory of elasticity for plates can be reduced to a two-dimensional one using the method of expanding the SSS into infinite mathematical series along the transverse coordinate using various variational principles. As obtained in [2, p. 51], Reissner's variational principle [57, p. 90] has certain advantages in determining the SSS of plates. This is the basis for the choice of Reissner's variational principle for the construction of a new variant of MT plates. According to Reissner's variational principle, the corresponding Reissner equation for elastic bodies (disregarding volume forces) has the following form:

$$\begin{gathered}\iiint(\sigma_x\delta\varepsilon_x+\sigma_y\delta\varepsilon_y+\sigma_z\delta\varepsilon_z+\sigma_{xy}\delta\gamma_{yx}+\sigma_{xz}\delta\gamma_{xz}+\sigma_{yz}\delta\gamma_{yz}+\\ +(\varepsilon_x-\varepsilon_x(\sigma_{ij}))\delta\sigma_x+(\varepsilon_y-\varepsilon_y(\sigma_{ij}))\delta\sigma_y+(\varepsilon_z-\varepsilon_z(\sigma_{ij}))\delta\sigma_z+\\ +(\gamma_{yx}-\gamma_{yx}(\sigma_{ij}))\delta\sigma_{yx}+(\gamma_{xz}-\gamma_{xz}(\sigma_{ij}))\delta\sigma_{xz}+\\ +(\gamma_{yz}-\gamma_{yz}(\sigma_{ij}))\delta\sigma_{yz})\,dx\,dy\,dz-\\ -\iint_A(X_\nu\delta U+Y_\nu\delta V+Z_\nu\delta W\quad dA=0,\end{gathered}\tag{2.1}$$

where A is the area of the face planes and the side surface of the plate; X_ν, Y_ν, Z_ν – intensity of component surface forces in projections on the coordinate axis; expressions with the sign δ mean the corresponding variations; $\sigma_x,...,\sigma_{yz,}\,\varepsilon_x,...,\gamma_{yz}$ – stress and strain components that depend on displacement components (3.9); integration in the triple integral is performed over the entire volume, and in the double integral over the entire A surface. The rectangular Cartesian coordinate system $Oxyz$ is introduced as follows: Ox, Oy axes are directed in the middle plane (in the plane of isotropy); the Oz axis is directed perpendicular to the median plane upwards.

On the upper and lower front faces of the plate, a static transverse load $q_1(x,y)$ and $q_2(x,y)$, respectively, directed downwards acts. The boundary conditions on the front faces have the following form

$$\sigma_z(z=\pm\frac{h}{2})=\frac{1}{2}(\mp q(x,y)-p(x,y));\sigma_{xz}(z=\pm\frac{h}{2})=\sigma_{yz}(z=\pm\frac{h}{2})=0,\tag{2.2}$$

where $p(x,y)/2$ and $q(x,y)/2$ are symmetric and obliquely symmetric relative to the median plane transverse loads acting on the front faces of the plate:

$$p(x,y)=q_1(x,y)-q_2(x,y)\,,\ q(x,y)=q_1(x,y)+q_2(x,y)\,.$$

2.3. Components of SSS. Let us present the SSS components, considering them as functions of three variables, which are developed into infinite mathematical series by Legendre polynomials from the transverse coordinate z.

2.3.1. Components of displacements. We develop the components of displacements $U(x,y,z), V(x,y,z)$ (tangential movements) and $W(x,y,z)$ (transverse movements) into infinite mathematical series by Legendre polynomials from the transverse coordinate. We take the partial sums of these series – approximation K0-N (N is an odd natural number). In the K0-N approximation, components (multipliers for Legendre polynomials) with indices 0,1, N: $u_0, v_0, u_1, v_1, w_1, ..., u_N, v_N, w_N$ are taken into account in the mathematical series for tangential movements in partial sums. Partial sums have the form:

$$U(x,y,z)=\sum_{k=0}^{N} P_k(2z/h)\,u_k(x,y),(U,u \to V,v); \tag{2.3}$$

$$W(x,y,z)=\sum_{k=1}^{N} P_{k-1}(2z/h)\,w_k(x,y),$$

where $P_k(2z/h)$ are the Legendre polynomials of the transverse coordinate z.

2.3.2. Image of stresses in the plate. Stress components are also obtained in the form of infinite mathematical series by Legendre polynomials. They have the following form, taking into account (2.1)–(2.3):

$$\sigma_{xz}(x,y,z)=\sum_{n=0}^{N+1} P_n(2z/h)\,t_{xn}(x,y);\ \sigma_{yz}(x,y,z)=\sum_{n=0}^{N+1} P_n(2z/h)\,t_{yn}(x,y); \tag{2.4}$$

$$\sigma_z(x,y,z)=\sum_{n=0}^{N+2} P_n(2z/h)\,s_{zn}(x,y);$$

$$\sigma_x(x,y,z)=\sum_{n=0}^{N+2} P_n(2z/h)\,s_{xn}(x,y),\ (x,y);$$

$$\sigma_{xy}(x,y,z)=\sum_{n=0}^{N} P_n(2z/h)\,t_{yxn}(x,y),$$

where

$$t_{xn}(x,y)=\sum_{i=1,3}^{N} (h_{0ni} w_{i,x} + l_{0ni} u_i),\ (n=0,2,...,N+1); \tag{2.5}$$

$$t_{xn}(x,y)=\sum_{i=2,4}^{N-1} (h_{0ni} w_{i,x} + l_{0ni} u_i),\ (n=1,3,...,N);$$

$$s_{zn}(x,y)=\sum_{i=2,4}^{N-1} p_{ni}w_i+\sum_{i=0,2}^{N-1} g_{ni}\phi_i+g_{np}p\,,\ (n=0,2,...,N+1);$$

$$s_{zn}(x,y)=\sum_{i=3,5}^{N} p_{ni}\,w_i+\sum_{i=1,3}^{N} g_{ni}\,\phi_i+g_{nq}\,q\,,\ (n=1,3,...,N+2);$$

$$s_{xn}(x,y)=d_0(u_{n,x}+\nu\, v_{n,y})+d_{10}s_{zn}\,,\ (n=0,1,...,N)\,;$$

$$s_{xn}(x,y)=d_{10}\,s_{zn}\,,\ (n=N+1,N+2)\,,\ (x,y;\,u_k,v_k);$$

$$(\nabla^2-m_1)(\nabla^2-m_2)F_{1\Pi}=0\,;\ \phi_i=u_{i,x}+v_{i,y}\,,$$

h, l, p, g with indices are mechanical and geometrical parameters (MGP); d_0, d_{10}, G – mechanical constants of the plate material.

It is important to note that the transverse stresses exactly satisfy the boundary conditions (2.2). This significantly increases the accuracy of this variant of MT and distinguishes it from many other variants.

The equilibrium equation and the boundary conditions on the circuit follow from Reissner's variational equation (2.1).

The system of differential equations for skew-symmetric and symmetric deformations in the K0-N approximation are given in [49, p. 229], and the boundary conditions are in [49, p. 230].

For the constructed variant of the MT of plates, with the increase in the number of terms in the partial sums of the development of the SSS components into mathematical series, the order of systems of differential equations of equilibrium and the mathematical complexity of their solution also increases. Therefore, there is a problem in solving such systems. This work is devoted to the solution of this problem, which is ideologically a continuation of [49, p. 223] work.

3. Solving the integration problem systems of high-order equilibrium equations

3.1. Methodology of integration of initial systems of equilibrium equations. Let us consider skew-symmetric deformation here and in the future. The methodology is given in [49, p. 221].

In Sections 3.2, 3.3, we present some systems of equations with [49, p. 221] for a better understanding of the further solution of boundary value problems.

Consider the system of differential equations (2.7) [49, p. 229] in the K13...N approximation, which describes skew-symmetric deformation. We will refer in paragraphs 3.2 – 3.4 to the relevant equations of the work [49, pp. 235, 236, 237].

3.2. Transformed systems of differential equations. These are systems of differential equations (4.1a), (4.1.b) [49, p. 235] for the vortex edge effect, and (4.2), (4.3) [49, p. 236] for the internal SSS with a potential edge effect.

3.3. Deterministic systems of differential equations. This is the differential equation (4.5) for the vortex edge effect [49, p. 236] and the system of equations (4.9) [49, p. 237] for the internal SSS with the potential edge effect.

3.4. General solutions of transformed and deterministic systems of differential equations. These are solutions (4.4) and (4.6) [49, p. 236] for the vortex edge effect and (4.8), (4.10a), (4.10b) [49, p. 237] for the internal SSS with a potential edge effect.

In the future, we will consider circular (circular and annular) transversally isotropic plates of arbitrary constant thickness in the K13 approximation (skew-symmetric deformation).

4. Axisymmetric bending of circular transtropical plates under skew-symmetric deformation

4.1. Main dependencies. Axisymmetric deformation in the K13 approximation of a circular transtropical plate (circular with radius a or ring with radii $b, a; b \langle a$) of thickness h is considered. A cylindrical coordinate system (r,θ,z) is introduced. Tangential coordinates in the middle plane, axis z is perpendicular to the middle plane, directed upwards. Boundary conditions on the front planes:

$$\sigma_z(z=\pm h/2)=\mp q(r)/2;\ \ \sigma_{rz}(z=\pm h/2)=0\ ;\ \sigma_{\theta z}(z=\pm h/2)=0\ , \quad (4.1)$$

where $q(r)/2$ is a transverse axisymmetric load.

Dependencies for axisymmetric deformation:

$$\varepsilon_r=(\sigma_r-v\sigma_\theta)/E-v'\sigma_z/E',\ (r,\theta);\ \varepsilon_z=(\sigma_z-v'(\sigma_r+\sigma_\theta))/E';$$
$$\gamma_{rz}=\sigma_{rz}/G';$$
$$\varepsilon_r=\partial U/\partial r;\ \ \varepsilon_\theta=U/r;\ \ \varepsilon_z=\partial W/\partial z;\ \ \gamma_{rz}=\partial W/\partial r+\partial U/\partial z,$$
$$\gamma_{r\theta}=\gamma_{z\theta}=0,$$

where $U, W,\ \varepsilon_r, \varepsilon_\theta, \varepsilon_z, \gamma_{rz},\ \sigma_r, \sigma_\theta, \sigma_z, \sigma_{rz}$ are functions from r, z. The constants included in these dependencies are generally accepted.

SSS components based on (2.4) and (2.5) are represented in the form of the following partial sums of series:

$$U\ (r,z) = \sum_{k=1,3,...}^{3} P_k(2z/h)u_k(r);\ V=0;\ W(r,z) = \sum_{k=1,3,...}^{3} P_{k-1}(2z/h)w_k(r);\ \ (4.2)$$

$$\sigma_r(r,z) = \sum_{k=1,3}^{5} P_k(2z/h)s_{rk}(r),\ \ (r,\theta)\ ;\ \sigma_z(r,z) = \sum_{k=1,3,...}^{5} P_k(2z/h)s_{zk}(r);$$

$$\sigma_{rz}(r,z) = \sum_{k=0,2,...}^{4} P_k(2z/h)t_{rk}(r),\ \ \sigma_{r\theta} = \sigma_{z\theta} = 0,$$

where, taking into account the components u_1, u_3, w_1, w_3 ($k = 1;\ 3$), we have:

$$s_{z1} = -3q/5 - 3\omega_3/70;\ \ s_{z3} = q/10 + \omega_3/15;\ \ s_{z5} = -\omega_3/42;\ \ (4.3)$$

$$s_{r1}(r) = d_0u_1' + d_0 v u_1/r + d_{10}s_{z1};\ \ s_{r3}(r) = d_0u_3' + d_0 v u_3/r + d_{10}s_{z3};$$
$$s_{r5}(r) = d_{10}s_{z5};$$

$$s_{\theta 1}(r) = d_0 v u_1' + d_0\, u_1/r + d_{10}s_{z1};\ \ s_{\theta 3}(r) = d_0 v u_3' + d_0u_3/r + d_{10}s_{z3};$$

$$s_{\theta 5}(r) = d_{10}s_{z5};$$

$$t_{r0} = Q_{1r}/h;\ \ t_{r2} = -Q_{1r}/h + 3Q_{3r}/(7h);\ \ t_{r4} = -3Q_{3r}/(7h);$$

$$Q_{kr}(r) = h_{k1}w_1' + h_{k3}w_3' + l_{k1}u_1 + l_{k3}u_3,\ (k=1,3)\ ;$$

$$\omega_3(r) = q_{33}w_3 + e_{31}\phi_1 + e_{33}\phi_3 + e_{3q}q\ ;\ \phi_k(r) = u_k' + u_k/r\ .$$

In relations (4.3):

$$l_{11} = 28G'/15;\ l_{11} = 6G'/5;\ l_{31} = 14G'/5;\ l_{33} = 84G'/5\ ;$$
$$h_{k1},\ h_{k3},\ q_{33},\ ...,\ e_{3q}\ ;\qquad (4.4)$$

$$h_{11} = 14G'h/15;\ h_{13} = -G'h/15;\ l_{11} = 28G'/15;\ l_{13} = 6G'/5;$$
$$h_{22} = 7G'h/6;\ l_{22} = 7G';\ h_{31} = h_{33} = 7G'h/5;\ \ l_{31} = 14G'/5;\ l_{33} = 84G'/5;$$

$$q_{22} = -14/(hd_{20});\ e_{20} = -7d_{30};\ e_{22} = 2d_{30};\ e_{2p} = -7/2; q_{33} = -66/(h\,d_{20});$$
$$e_{31} = -11d_{30};\ e_{33} = 22d_{30}/3;\ e_{3q} = -22/3;\ d_{20} = (1-2d_{10}\nu')/E'; d_{30} = d_{10}/d_{20}.$$

$$G = E/(2(1+v));\ d_0 = E/(1-v^2),\ \ d_{10} = E v'/(E'(1-v))\ .$$

Boundary conditions (4.1) are fulfilled exactly.

4.2. A system of differential equations of equilibrium. The system of differential equations of equilibrium of the internal SSS with a potential marginal effect in polar coordinates has the form:

$$\beta_{1k3}u_1+\beta_{k33}u_3+\beta_{1k1}\phi_1'+\beta_{k31}\phi_3'+\beta_{k51}w_1'+\beta_{k61}w_3'=\beta_{uk}\,q',\ (k=1,3);\ (4.5)$$

$$\beta_{151}\phi_1+\beta_{351}\phi_3+\beta_{551}\nabla^2 w_1+\beta_{561}\nabla^2 w_3=\beta_{w1}\,q;$$

$$\beta_{161}\phi_1+\beta_{361}\phi_3+\beta_{561}\nabla^2 w_1+(\beta_{661}\nabla^2+\beta_{662})\,w_3=\beta_{w3}\,q,$$

$$(\nabla^2=d^2/d\,r^2+d/dr/r),$$

where $q=q(r)$, ∇^2 is the Laplace operator; β with indices – MGP:

$$\beta_{111}=\frac{h}{3}(d_0-\frac{3}{70}d_{10}e_{31});\ \beta_{112}=\frac{hG}{3};\ \beta_{113}=-\frac{2}{h}l_{11};\qquad(4.6)$$

$$\beta_{121}=\frac{h}{3}(d_0v+G-\frac{3}{70}d_{10}e_{31});\ \beta_{131}=-\frac{h}{70}d_{10}e_{33};\ \beta_{133}=-\frac{2}{h}l_{13};$$

$$\beta_{141}=-\frac{h}{70}d_{10}e_{33};\ \beta_{151}=-\frac{2}{h}h_{11};\ \beta_{161}=-(\frac{2}{h}h_{13}+\frac{hd_{10}q_{33}}{70});\ \beta_{u1}=\frac{2}{21}hd_{10};$$

$$\beta_{221}=\frac{hG}{3};\ \beta_{222}=\beta_{111};\ \beta_{223}=\beta_{113};\ \beta_{231}=\beta_{131};\ \beta_{242}=\beta_{141}=\beta_{131};$$

$$\beta_{243}=\beta_{133};\ \beta_{251}=\beta_{151};\ \beta_{331}=\frac{h}{7}(d_0+\frac{1}{15}d_{10}e_{33});\ \beta_{332}=\frac{hG}{7};$$

$$\beta_{333}=-\frac{6}{7h}l_{33};\ \beta_{341}=\frac{h}{7}(d_0v+G+\frac{1}{15}d_{10}e_{33});\ \beta_{351}=-\frac{6}{7h}h_{31};$$

$$\beta_{361}=\frac{1}{105}hd_{10}q_{33}-\frac{6}{7h}h_{33};\ \beta_{u3}=\frac{1}{18}hd_{10};\ \beta_{441}=\beta_{332};\ \beta_{442}=\beta_{331};$$

$$\beta_{443}=\beta_{333};\ \beta_{451}=\beta_{351};\ \beta_{461}=\beta_{361};\ \beta_{551}=-h_{11};\ \beta_{561}=-h_{13};$$

$$\beta_{w1}=-1;\ \beta_{661}=\frac{1}{35}(7h_{13}-3h_{33});\ \beta_{662}=-\frac{3q_{33}}{35};\ \beta_{w3}=-\frac{3}{7}.$$

Functions $\phi_j(r)$ and $u_j(r)$ are from the system (4.5):

$$\phi_j(r)=\lambda_{j1}\nabla^2 w_1+\lambda_{j2}\,w_3+\lambda_{j3}\nabla^2 w_3+\lambda_{j4}\,q;\qquad(4.7)$$

$$u_j(r)=\lambda_{j\phi1}\,\phi_1'+\lambda_{j\phi3}\,\phi_3'+\lambda_{jw1}\,w_1'+\lambda_{jw3}\,w_3'+\lambda_{jq}\,q',\ (j=1,3),$$

where constants λ with subscripts are mechanical and geometrical parameters:

$$\lambda_{11}=(\beta_{351}\beta_{561}-\beta_{361}\beta_{551})/\Delta;\ \lambda_{12}=\beta_{351}\beta_{662}/\Delta;$$

$$\lambda_{13}=(\beta_{351}\beta_{661}-\beta_{361}\beta_{561})/\Delta;\qquad(4.8)$$

$$\lambda_{14} = (\beta_{361}\beta_{w1} - \beta_{351}\beta_{w3})/\Delta;\ \ \lambda_{31} = (\beta_{161}\beta_{551} - \beta_{151}\beta_{561})/\Delta;$$

$$\lambda_{32} = -\beta_{151}\beta_{662}/\Delta;$$

$$\lambda_{33} = (\beta_{161}\beta_{561} - \beta_{151}\beta_{661})/\Delta;\ \lambda_{34} = (\beta_{151}\beta_{w3} - \beta_{161}\beta_{w1})/\Delta;$$

$$\Delta = \beta_{151}\beta_{361} - \beta_{161}\beta_{351}.$$

$$\lambda_{1\phi 1} = (\beta_{133}\beta_{131} - \beta_{333}\beta_{111})/\Delta_u;\ \lambda_{1\phi 3} = (\beta_{133}\beta_{331} - \beta_{333}\beta_{131})/\Delta_u;$$

$$\lambda_{1\psi 1} = -\beta_{333}\beta_{112}/\Delta_u;$$

$$\lambda_{1\psi 3} = \beta_{133}\beta_{332}/\Delta_u;\ \lambda_{1w1} = (\beta_{133}\beta_{351} - \beta_{333}\beta_{151})/\Delta_u;$$

$$\lambda_{1w3} = (\beta_{133}\beta_{361} - \beta_{333}\beta_{161})/\Delta_u;$$

$$\lambda_{1q} = (\beta_{333}\beta_{u1} - \beta_{133}\beta_{u3})/\Delta_u;\ \lambda_{3\phi 1} = (\beta_{133}\beta_{111} - \beta_{113}\beta_{131})/\Delta_u$$

$$\lambda_{3\phi 3} = (\beta_{133}\beta_{131} - \beta_{113}\beta_{331})/\Delta_u;\ \lambda_{3\psi 1} = \beta_{133}\beta_{112}/\Delta_u;\ \lambda_{3\psi 3} = -\beta_{113}\beta_{332}/\Delta_u;$$

$$\lambda_{3w1} = (\beta_{133}\beta_{151} - \beta_{113}\beta_{351})/\Delta_u;\ \lambda_{3w3} = (\beta_{133}\beta_{161} - \beta_{113}\beta_{361})/\Delta_u;$$

$$\lambda_{3q} = (\beta_{113}\beta_{u3} - \beta_{133}\beta_{u1})/\Delta_u;\ \Delta_u = \beta_{113}\beta_{333} - \beta_{133}^2.$$

4.3. Transformed system of differential equations. The system of equations (4.5) is reduced to a system of two equations with respect to $w_1(r)$ and $w_3(r)$:

$$\sum_{i=1,3}^{3} \Pi_{ji} w_i(r) = \Pi_{jq} q(r) \quad (j = 1,3), \tag{4.9}$$

where Π_{ji} are fourth-order differential operators. Π_{jq} – second-order differential operators. In the approximation K13 (N = 3), the differential operators of equations (4.9) Π_{ji} $(i, j = 1,3)$ and Π_{jq} have the following form:

$$\Pi_{11} = \mu_{114}\nabla^4 + \mu_{112}\nabla^2;\ \Pi_{13} = \mu_{134}\nabla^4 + \mu_{32}\nabla^2 + \mu_{130}; \tag{4.10}$$

$$\Pi_{31} = \mu_{314}\nabla^4 + \mu_{312}\nabla^2;\ \Pi_{33} = \mu_{334}\nabla^4 + \mu_{332}\nabla^2 + \mu_{330};$$

$$\Pi_{1q} = \mu_{12}\nabla^2 - \mu_{10};\ \Pi_{3q} = \mu_{32}\nabla^2 - \mu_{30},$$

where $\mu_{114}, \mu_{112}, ..., \mu_{30}$ – MGP:

$$\mu_{114} = \beta_{111}\lambda_{11} + \beta_{131}\lambda_{31};\ \ \mu_{134} = \beta_{111}\lambda_{13} + \beta_{131}\lambda_{33}; \tag{4.11}$$

$$\mu_{132} = \beta_{113}\lambda_{13} + \beta_{111}\lambda_{12} + \beta_{133}\lambda_{33} + \beta_{131}\lambda_{32} + \beta_{161};$$

$$\mu_{12} = \beta_{u1} - (\beta_{111}\lambda_{14} + \beta_{131}\lambda_{34});\ \ \mu_{10} = \beta_{113}\lambda_{14} + \beta_{133}\lambda_{34};$$

$$\mu_{314} = \beta_{131}\lambda_{11} + \beta_{331}\lambda_{31};\;\; \mu_{312} = \beta_{133}\lambda_{11} + \beta_{333}\lambda_{31} + \beta_{351};$$

$$\mu_{334} = \beta_{131}\lambda_{13} + \beta_{331}\lambda_{33};\;\; \mu_{332} = \beta_{133}\lambda_{13} + \beta_{131}\lambda_{12} + \beta_{333}\lambda_{33} + \beta_{331}\lambda_{32} + \beta_{361};$$

$$\mu_{330} = \beta_{133}\lambda_{12} + \beta_{333}\lambda_{32};\;\; \mu_{32} = \beta_{u3} - (\beta_{131}\lambda_{14} + \beta_{331}\lambda_{34});$$

$$\mu_{30} = \beta_{133}\lambda_{14} + \beta_{333}\lambda_{34}.$$

4.4. Deterministic system of differential equations. The deterministic system of differential equations, which describes the internal SSS with a potential marginal effect, is obtained on the basis of (4.9), (4.10) and has the form:

$$D_0\, D_0\, D_1\, D_2 \cdot \varPhi_k\,(r) = a_{k\,0}\, D_{k\,0}\, q(r)\,, \quad k = 1,3\,, \tag{4.12}$$

where

$$D_0 = \nabla^2;\; D_i = \nabla^2 - s_i;\, D_{k\,0} = \nabla^2 - s_{k\,0};\;\; i = 1,2,...,N-1;$$

s_i, s_{k0}, a_{k0} – MGP; s_i – the roots of the corresponding characteristic equation (for transversely isotropic plates with small shear stiffness $s_i \rangle\, 0$); of functions $\varPhi_k\,(x,y)$ are sought.

4.5. Solutions of the transformed system of differential equations (4.9). Forms of general solutions of system (4.9) are obtained by the operator method and have the form:

$$w_i\,(r) = \sum_{k=1,3}^{3} \varPi^0_{k\,i}\varPhi_k\,(r), (i = 1,\, 3;\; N = 3), \tag{4.13}$$

where $\varPi^0_{k\,i}$ are the appendages of the determinant of the $\varPi_0$ system (4.9), $\varPhi_k\,(x,y)$ are the new sought functions that are determined from (4.12).

4.6. Axisymmetric load on the ring. Partial and general solutions of system (4.12). Let's consider the load of the plate on the annular area. Consider a plate under the action of a uniformly distributed load q_0 along a circular ring with radii r_1 and r_2 $(r_1 \langle\, r_2)$:

$$q(r) = \begin{cases} 0, & (r \,\langle\, r_1\,); \\ q_0, & (r_1 \,\langle\, r \,\langle\, r_2\,); \\ 0, & (r \rangle\, r_2\,). \end{cases} \tag{4.14}$$

4.6.1. Partial solutions of the system (4.12). Partial solutions $\varPhi_{k\,r}\,(r)$ of the defining differential equations (4.12) p according to (7.2)–(7.4) [49, p. 245] have this form:

$$\Phi_{kr}(r) = -\frac{a_{k0}q_0}{s_1\, s_2}\big(\frac{s_2}{s_1 s_{12}\sqrt{s_1}}(s_1 - s_{k0})I_0(r\sqrt{s_1})(r_1K_1(r_1\sqrt{s_1}) - r_2K_1(r_2\sqrt{s_1})) +$$

$$+\frac{s_1}{s_2 s_{21}\sqrt{s_2}}(s_2 - s_{k0})I_0(r\sqrt{s_2})(r_1K_1(r_1\sqrt{s_2}) - r_2K_1(r_2\sqrt{s_2})) - \frac{1}{4}(r_2^2 - r_1^2) +$$

$$+\frac{1}{2}(\frac{s_{k0}r^2}{4} - 1 - s_{k0}s_{12}^0)(r_2^2\ln r_2 - r_1^2\ln r_1) + \frac{s_{k0}}{16}(r_2^4\ln r_2 - r_1^4\ln r_1 +$$

$$+4s_{12}^0(r_2^2 - r_1^2) - \frac{1}{4}(r_2^4 - r_1^4) + r^2(r_2^2 - r_1^2))), \quad (r\langle r_1,\ k = 1,3); \qquad (4.15)$$

$$\Phi_{kr}(r) = -\frac{a_{k0}q_0}{s_1\, s_2}\big(\frac{s_2}{s_1 s_{12}\sqrt{s_1}}(s_1 - s_{k0})K_0(r\sqrt{s_1})(r_2I_1(r_2\sqrt{s_1}) - r_1I_1(r_1\sqrt{s_1})) +$$

$$+\frac{s_1}{s_2 s_{21}\sqrt{s_2}}(s_2 - s_{k0})K_0(r\sqrt{s_2})(r_2I_1(r_2\sqrt{s_2}) - r_1I_1(r_1\sqrt{s_2})) -$$

$$-\frac{1}{2}(1 + \ln r + s_{k0}s_{12}^0\ln r)(r_2^2 - r_1^2) + \frac{s_{k0}}{16}(2r^2(r_2^2 - r_1^2)\ln r + (r_2^4 - r_1^4)(1 + \ln r))),$$

$$(r\rangle r_2, k = 1,3); \qquad (4.16)$$

$$\Phi_{kr}(r) = -\frac{a_{k0}q_0}{s_1 s_2}(A_{kr}(r) + B_{kr}(r))\,; \; (4.17)$$

$$A_{kr}(r) = \frac{s_2}{s_1 s_{12}\sqrt{s_1}}(s_1 - s_{k0})I_0(r\sqrt{s_1})(r\,K_1(r\,\sqrt{s_1}) - r_2K_1(r_2\sqrt{s_1})) +$$

$$+\frac{s_1}{s_2 s_{21}\sqrt{s_2}}(s_2 - s_{k0})I_0(r\sqrt{s_2})(r\,K_1(r\,\sqrt{s_2}) - r_2K_1(r_2\sqrt{s_2})) +$$

$$+\frac{1}{4}(r_2^2 - r^2)(s_{k0}s_{12}^0 - 1) + \frac{1}{2}(\frac{s_{k0}r^2}{4} - 1 - s_{k0}s_{12}^0)(r_2^2\ln r_2 - r^2\ln r);$$

$$B_{kr}(r) = \frac{s_{k0}}{16}(r_2^4\ln r_2 - r^4\ln r - \frac{1}{4}(r_2^4 - r^4) + r^2(r_2^2 - r^2)) +$$

$$+\frac{s_2}{s_1 s_{12}\sqrt{s_1}}(s_1 - s_{k0})K_0(r\sqrt{s_1})(r\,I_1(r\,\sqrt{s_1}) - r_1I_1(r_1\sqrt{s_1})) +$$

$$+\frac{s_1}{s_2 s_{21}\sqrt{s_2}}(s_2 - s_{k0})K_0(r\sqrt{s_2})(r\,I_1(r\,\sqrt{s_2}) - r_1I_1(r_1\sqrt{s_2})) -$$

$$-\frac{1}{2}(1 + \ln r + s_{k0}s_{12}^0\ln r)(r^2 - r_1^2) + \frac{s_{k0}}{16}(2r^2(r^2 - r_1^2)\ln r + (r^4 - r_1^4)(1 + \ln r)),$$

$$(r_1\langle r\langle r_2),$$

where I_1, K_1 is a modified first-order Bessel and Macdonald functions.

The conjugation conditions for $r = r_1$ and $r = r_2$ are satisfied.

4.6.2. General solutions of system (4.12). The general solutions of the defining differential equations (4.12) have the following form:

$$\Phi_1(r) = A_0 + B_0 r^2 + C_0 \ln r + D_0 r^2 \ln r + A_1 I_0(r\sqrt{s_1}) + B_1 K_0(r\sqrt{s_1}) + \quad (4.18)$$

$$+ A_2 I_0(r\sqrt{s_2}) + B_2 K_0(r\sqrt{s_2}) + \Phi_{1r}(r);\ \Phi_3(r) = \Phi_{3r}(r).$$

Without stopping at cumbersome calculations, we will give general solutions for the SSS components from the uniform loading of the plate over the annular region.

5. General solutions for SSS components

The general solutions for the SSS components for circular and annular plates from skew-symmetric loading on the annular region (4.14) are determined taking into account (4.2)–(4.4), (4.6)–(4.8), (4.11), (4.13)–(4.18).

5.1. General solutions for displacements. Transverse displacements:

$$W(r,z) = \sum_{k=1,3}^{3} P_k(2z/h)\, w_k(r)\,. \quad (5.1)$$

The components of displacements are determined from the following dependencies:

for $r \langle r_1$

$$w_1(r) =$$

$$= (A_0\mu_{330} + 4\mu_{332}(B_0 + D_0) + B_0\mu_{330}r^2 + (4D_0\mu_{332} + C_0\mu_{330})\ln r + D_0\mu_{330}r^2 \ln r +$$

$$+ \alpha_{101}A_{11}(r\sqrt{s_1}) + \alpha_{102}A_{22}(r\sqrt{s_2})) + q_0(k_{1r1}I_0(r\sqrt{s_1}) + k_{1r2}I_0(r\sqrt{s_2}) + k_{1r3}r^2 + k_{1r0});$$

$$w_3(r) = (-4\mu_{312}(B_0 + D_0) - 4D_0\mu_{312}\ln r + \alpha_{301}A_{11}(r\sqrt{s_1}) + \alpha_{302}A_{22}(r\sqrt{s_2})) +$$

$$+ q_0(k_{3r1}I_0(r\sqrt{s_1}) + k_{3r2}I_0(r\sqrt{s_2}) + k_{3r0}), \quad (5.2)$$

where

$$\alpha_{10i} = \mu_{334}s_i^2 + \mu_{332}s_i + \mu_{330},\ \alpha_{30i} = -s_i(\mu_{314}s_i + \mu_{312}); \quad (5.3)$$

$$k_{1ri} = c_{1i}(a_1 a_{1i1}(\mu_{334}s_i^2 + \mu_{332}s_i + \mu_{330}) - a_3 a_{1i3}s_i(\mu_{134}s_i + \mu_{132}));$$

$$k_{3ri} = c_{1i}\, s_i(a_3 a_{1i3}\mu_{114}s_i - a_1 a_{1i1}(\mu_{314}s_i + \mu_{312})),\ (i = 1,2);$$

$$k_{1r0} = a_1(4\mu_{332}a_{131}b_{11} + \mu_{330}(a_{141}b_{12} + a_{151}b_{13} + a_{161}b_{14}) - 4a_3\mu_{132}a_{133}b_{11};$$

$$c_{1i} = r_1 K_1(r_1\sqrt{s_i}) - r_2 K_1(r_2\sqrt{s_i}),\ (i = 1,2);\ a_1 = -\frac{a_{k0}}{s_1 s_2};\ a_{11k} = \frac{s_2(s_1 - s_{k0})}{s_1 s_{12}\sqrt{s_1}};$$

$$a_{12k}=\frac{s_1(s_2-s_{k0})}{s_2s_{21}\sqrt{s_2}};a_{13k}=\frac{s_{k0}}{8};\ a_{14k}=-\frac{1}{2}(1+s_{k0}s_{12}^0);a_{15k}=\frac{s_{k0}}{16};\ a_{16k}=s_{k0}s_{12}^0-1;$$

$$b_{11}=r_2^2\ln r_2-r_1^2\ln r_1+(r_2^2-r_1^2)/2,\ b_{12}=r_2^2\ln r_2-r_1^2\ln r_1;$$

$$b_{13}=r_2^4\ln r_2-r_1^4\ln r_1-(r_2^4-r_1^4)/4,\ b_{14}=(r_2^2-r_1^2)/4;$$

$$A_{11}(r\sqrt{s_1})=A_1I_0(r\sqrt{s_1})+B_1K_0(r\sqrt{s_1}),\ A_{22}(r\sqrt{s_2})=A_2I_0(r\sqrt{s_2})+B_2K_0(r\sqrt{s_2});$$

for $r\rangle r_2$

$$w_1(r)=(A_0\mu_{330}+4\mu_{332}(B_0+D_0)+B_0\mu_{330}r^2+(4D_0\mu_{332}+C_0\mu_{330})\ln r+ \quad (5.4)$$
$$+D_0\mu_{330}r^2\ln r+\alpha_{101}A_{11}(r\sqrt{s_1})+\alpha_{102}A_{22}(r\sqrt{s_2}))+$$
$$+q_0(m_{1r1}K_0(r\sqrt{s_1})+m_{1r2}K_0(r\sqrt{s_2})+m_{1r}\ln r+m_{1r3}r^2\ln r+m_{1r0});$$
$$w_3(r)=(-4\mu_{312}(B_0+D_0)-4D_0\mu_{312}\ln r+\alpha_{301}A_{11}(r\sqrt{s_1})+\alpha_{302}A_{22}(r\sqrt{s_2}))+$$
$$+q_0(m_{3r1}K_0(r\sqrt{s_1})+m_{3r2}K_0(r\sqrt{s_2})+m_{3r}\ln r+m_{3r0}),$$

where

$$m_{1ri}=c_{2i}(a_1a_{1i1}(\mu_{334}s_i^2+\mu_{332}s_i+\mu_{330})-a_3a_{1i3}s_i(\mu_{134}s_i+\mu_{132}));$$
$$m_{3ri}=c_{2i}s_i(a_3a_{1i3}\mu_{114}s_i-a_1a_{1i1}(\mu_{314}s_i+\mu_{312})),\ (i=1,2);$$
$$m_{1r0}=a_1(4\mu_{332}c_{221}+\mu_{330}c_{231})-4a_3\mu_{132}c_{223};$$
$$m_{3r0}=-4a_1\mu_{312}c_{221},\ m_{1r3}=a_1\mu_{330}c_{221};$$
$$m_{1r}=a_1(4\mu_{332}c_{221}+\mu_{330}c_{211})-4a_3\mu_{132}c_{223},\ m_{3r}=-4a_1\mu_{312}c_{221};$$

for $r_1\langle r\langle r_2$

$$w_1(r)=(A_0\mu_{330}+4\mu_{332}(B_0+D_0)+B_0\mu_{330}r^2+(4D_0\mu_{332}+C_0\mu_{330})\ln r+ \quad (5.5)$$
$$+D_0\mu_{330}r^2\ln r+\alpha_{101}A_{11}(r\sqrt{s_1})+\alpha_{102}A_{22}(r\sqrt{s_2}))+$$
$$+(q_0(n_{101}(r_2K_1(r_2\sqrt{s_1})I_0(r\sqrt{s_1})+r_1I_1(r_1\sqrt{s_1})K_0(r\sqrt{s_1}))+$$
$$+n_{102}(r_2K_1(r_2\sqrt{s_2})I_0(r\sqrt{s_2})+r_1I_1(r_1\sqrt{s_2})K_0(r\sqrt{s_2}))+$$
$$+a_1\mu_{330}(a_{111}\phi_{s1r}+a_{121}\phi_{s2r})+n_{1r}\ln r+n_{1r1}r^2\ln r+n_{1r2}r^2+n_{103}r^4+n_{1r0}));$$
$$w_3(r)=(-4\mu_{312}(B_0+D_0)-4D_0\mu_{312}\ln r+\alpha_{301}A_{11}(r\sqrt{s_1})+\alpha_{302}A_{22}(r\sqrt{s_2}))+$$
$$+q_0(n_{301}(r_2K_1(r_2\sqrt{s_1})I_0(r\sqrt{s_1})+r_1I_1(r_1\sqrt{s_1})K_0(r\sqrt{s_1}))+$$

$$+n_{302}(r_2K_1(r_2\sqrt{s_2})I_0(r\sqrt{s_2})+r_1I_1(r_1\sqrt{s_2})K_0(r\sqrt{s_2}))+n_{3r}\ln r+n_{303}r^2+n_{3r0})\ ,$$

where

$$n_{10i}=s_i(a_3a_{1i3}(\mu_{134}s_i+\mu_{132})-a_1a_{1i1}(\mu_{334}s_i+\mu_{332}));$$

$$n_{30i}=s_i(a_1a_{1i1}(\mu_{314}s_i+\mu_{312})-a_3a_{1i3}\mu_{114}s_i),\ (i=1,2);$$

$$n_{1r}=a_1(4\mu_{332}c_{311}+\mu_{330}c_{321})-4a_3\mu_{132}c_{313},\ n_{3r}=-4a_1\mu_{312}c_{311};$$

$$n_{1r0}=a_1(64\mu_{334}a_{311}+4\mu_{332}(c_{311}+c_{331})+\mu_{330}c_{341})-4a_3(16\mu_{134}a_{313}+\mu_{132}(c_{313}+c_{333});$$

$$n_{1r1}=a_1\mu_{330}c_{311}\ ,\ n_{1r2}=a_1(16\mu_{332}a_{311}+\mu_{330}c_{331})-16a_3\mu_{132}a_{313}\ ,\ n_{103}=a_1\mu_{330}c_{311}\ ;$$
$$n_{303}=-16a_1\mu_{312}a_{311},\ n_{3r0}=4(16a_3\mu_{114}a_{313}-a_1(16\mu_{314}a_{311}+\mu_{312}(c_{311}+c_{331}))).$$

The tangential displacements of $U_r(r,z)$ are as follows:

$$U_r(r,z)=\sum_{k=1,3}^{3}P_k(2z/h)u_k(r)\ . \tag{5.6}$$

where $u_k(r)$, $\phi_k(r)$ are determined according to (4.7), and $q(r)$ – according to (4.14).

$\phi_k(r)$ functions have the following form:

for $r\langle r_1$

$$\phi_k(r)=\phi_{k0}(r)+q_0((s_1(\lambda_{k1}k_{1r1}+\lambda_{k3}k_{3r1})+\lambda_{k2}k_{3r1})I_0(r\sqrt{s_1})+ \tag{5.7}$$

$$+(s_2(\lambda_{k1}k_{1r2}+\lambda_{k3}k_{3r2})+\lambda_{k2}k_{3r2})I_0(r\sqrt{s_2})+(4\lambda_{k1}k_{1r3}+\lambda_{k2}k_{3r0})),\ (k=1,3);$$

$$\phi_{k0}(r)=$$

$$=4B_0(\lambda_{k1}\mu_{330}-\lambda_{k2}\mu_{312})+4D_0((\lambda_{k1}\mu_{330}-\lambda_{k2}\mu_{312})\ln r+(\lambda_{k1}\mu_{330}-\lambda_{k2}\mu_{312}))+$$
$$+(s_1(\lambda_{k1}\alpha_{101}+\lambda_{k3}\alpha_{301})+\lambda_{k2}\alpha_{301})(A_1I_0(r\sqrt{s_1})+B_1K_0(r\sqrt{s_1}))+$$
$$+(s_2(\lambda_{k1}\alpha_{102}+\lambda_{k3}\alpha_{302})+\lambda_{k2}\alpha_{302})(A_2I_0(r\sqrt{s_2})+B_2K_0(r\sqrt{s_2}));$$

for $r\rangle r_2$

$$\phi_k(r)=\phi_{k0}(r)+q_0((s_1(\lambda_{k1}m_{1r1}+\lambda_{k3}m_{3r1})+\lambda_{k2}m_{3r1})K_0(r\sqrt{s_1})+ \tag{5.8}$$
$$+(s_2(\lambda_{k1}m_{1r2}+\lambda_{k3}m_{3r2})+\lambda_{k2}m_{3r2})K_0(r\sqrt{s_2})+$$

$$+(4\lambda_{k1}m_{1r3}+\lambda_{k2}m_{3r0})+(4\lambda_{k1}m_{1r3}+\lambda_{k2}m_{3r})\ln r)),\ \ (k=1,3);$$

for $r_1\ \langle\ r\ \langle\ r_2$

$$\phi_k(r)=\phi_{k0}(r)+q_0(a_{\phi1}(r_2K_1(r_2\sqrt{s_1})I_0(r\sqrt{s_1})+r_1I_1(r_1\sqrt{s_1})K_0(r\sqrt{s_1}))+ \tag{5.9}$$

$$+a_{\phi2}(r_2K_1(r_2\sqrt{s_2})I_0(r\sqrt{s_2})+r_1I_1(r_1\sqrt{s_2})K_0(r\sqrt{s_2}))+c_{\phi r0}+c_{\phi r1}\ln r+a_{\phi0}r^2)\ ,$$

where

$$a_{\phi 0}=16\lambda_{k1}n_{103}+\lambda_{k2}n_{303});\ a_{\phi 1}=s_1(\lambda_{k1}(n_{101}-a_1a_{111}\mu_{330})+\lambda_{k3}n_{301})+\lambda_{k2}n_{301};$$

$$a_{\phi 2}=s_2(\lambda_{k1}(n_{102}-a_1a_{121}\mu_{330})+\lambda_{k3}n_{302})+\lambda_{k2}n_{302};$$

$$q_0,\ r\in(r_1,\ r_2)$$

Components $u_k(r)$ of radial displacements are determined as follows:

for $r\langle r_1$

$$u_k(r)=u_{k0}(r)+q_0(r_{11k}I_1(r\sqrt{s_1})+r_{12k}I_1(r\sqrt{s_2})+2k_{1r3}r),\qquad(5.10)$$

where

$$u_{k0}(r)=(C_0\mu_{330}\alpha_{k11}+4D_0(\mu_{330}\alpha_{k12}+\mu_{332}\alpha_{k11}-\mu_{312}\alpha_{k31}))r^{-1}+$$

$$+\mu_{330}\alpha_{k11}(2B_0+D_0)r+\ 2D_0\mu_{330}\alpha_{k11}r\ln r+t_{11k}(A_1I_1(r\sqrt{s_1})-B_1K_1(r\sqrt{s_1}))+$$

$$+t_{12k}(A_2I_1(r\sqrt{s_2})-B_2K_1(r\sqrt{s_2}));$$

$$t_{1ik}=\sqrt{s_i}(\alpha_{10i}(\alpha_{k12}s_i+\alpha_{k11})+\alpha_{30i}(\alpha_{k32}s_i+\alpha_{k31}));$$

$$r_{1ik}=\sqrt{s_i}(k_{1ri}(\alpha_{k12}s_i+\alpha_{k11})+k_{3ri}(\alpha_{k32}s_i+\alpha_{k31})),\ (i=1,2;k=1,3);$$

for $r\rangle r_2$

$$u_k(r)=u_{k0}(r)+q_0(r_{21k}K_1(r\sqrt{s_1})+r_{22k}K_1(r\sqrt{s_2})+r_{23k}r^{-1}+r_{24k}r\ln r+r_{25k}r),\ (5.11)$$

where

$$r_{2ik}=-\sqrt{s_i}(m_{1ri}(\alpha_{k12}s_i+\alpha_{k11})+m_{3ri}(\alpha_{k32}s_i+\alpha_{k31})),\ (i=1,2;\ k=1,3);$$

$$r_{23k}=4\alpha_{k12}m_{1r3}+\alpha_{k11}m_{1r1}-\alpha_{k31}m_{3r};\ r_{24k}=2\alpha_{k11}m_{1r3};\ r_{25k}=\alpha_{k11}m_{1r3};$$

for $r_1\ \langle\ r\ \langle\ r_2$

$$u_k(r)=u_{k0}(r)+q_0(s_{31k}(r_2K_1(r_2\sqrt{s_1})I_1(r\sqrt{s_1})-r_1I_1(r_1\sqrt{s_1})K_1(r\sqrt{s_1}))+\ (5.12)$$

$$+s_{32k}(r_2K_1(r_2\sqrt{s_2})I_1(r\sqrt{s_2})-r_1I_1(r_1\sqrt{s_2})K_1(r\sqrt{s_2}))+$$

$$+r_{31k}r^{-1}+r_{32k}r+r_{33k}r\ln r+r_{34k}r^3),$$

where

$$t_{3ik}=\sqrt{s_i}(s_i(\alpha_{k12}\alpha_{10i}+\alpha_{k32}\alpha_{30i})+\alpha_{k11}\alpha_{10i}+\alpha_{k31}\alpha_{30i});$$

$$s_{3ik}=\sqrt{s_i}((n_{10i}-a_1\alpha_{1i1}\mu_{330})(\alpha_{k12}s_i+\alpha_{k11})+n_{30i}(\alpha_{k32}s_i+\alpha_{k31}));$$

$$r_{31k}=4\alpha_{k12}n_{1r1}+\alpha_{k11}n_{1r}+\alpha_{k31}n_{3r};$$

$$r_{32k}=32\alpha_{k12}n_{103}+\alpha_{k11}(n_{1r1}+2n_{1r2})+2\alpha_{k31}n_{303};$$
$$r_{33k}=2\alpha_{k11}n_{1r1};\ r_{34k}=4\alpha_{k11}n_{103}.$$

The displacements of $U_r(r,z)$ are determined according to (5.6) taking into account (4.14), (5.7)–(5.12).

5.2. General solutions for stresses in plates. Stresses $\sigma_r(r,z)$ are determined according to (4.2), (4.3) as follows:

$$\sigma_r(r,z)=\sum_{i=1,3}^{5}P_i(2z/h)s_{ri}(r)\,, \tag{5.13}$$

where

$$s_{ri}(r)=a_{iu1}u_{1,r}+a_{iu3}u_{3,r}+a_{iw3}w_3+a_{iq}q,\ (i=1,3,5)\,; \tag{5.14}$$

$$a_{1u1}=d_0-3e_{31}d_{10}/70;\ a_{1u3}=-3e_{33}d_{10}/70;$$
$$a_{1w3}=-3q_{33}d_{10}/70;\ a_{1q}=-3d_{10}(1+e_{3q}/14)/5;$$

$$a_{3u1}=e_{31}d_{10}/15;\ a_{3u3}=d_0+e_{33}d_{10}/15;$$
$$a_{3w3}=q_{33}d_{10}/15;\ a_{3q}=(1/10+e_{3q}/15)\,d_{10};$$

$$a_{5u1}=-e_{31}d_{10}/42;\ a_{5u3}=-e_{33}d_{10}/42;\ a_{5w3}=-q_{33}d_{10}/42;\ a_{5q}=-e_{3q}d_{10}/42.$$

In (5.13), (5.14) the constants d,e,q with subscripts are determined directly through the constants of the transversally isotropic material, $q(r)$ corresponds to (4.14), the displacement components $w_k(r)$ and $u_k(r)$ are determined by the previous formulas.

Transverse tangential stresses $\sigma_{rz}(r,z)$ according to (4.2) are as follows:

$$\sigma_{rz}(r,z)=\sum_{i=0,2}^{4}P_i(2z/h)t_{ri}(r)\,, \tag{5.15}$$

where the components of $t_{ri}(r)$ are determined by formulas (4.3):

$$t_{ri}(r)=h_{0i1}w_{1,r}+h_{0i3}w_{3,r}+l_{0i1}u_1+l_{0i3}u_3,\ (i=0,2,4), \tag{5.16}$$

$$h_{001}=h_{11}/h;\ h_{003}=h_{13}/h;\ h_{021}=(3h_{31}/7-h_{11})/h;\ h_{023}=(3h_{33}/7-h_{13})/h;$$
$$h_{041}=-3h_{31}/7;\ h_{043}=-3h_{33}/7;\ l_{001}=l_{11}/h;\ l_{003}=l_{13}/h;$$
$$l_{021}=(3l_{31}/7-l_{11})/h;\ l_{023}=(3l_{33}/7-l_{13})/h;\ l_{041}=-3l_{31}/7;\ l_{043}=-3l_{33}/7\,;$$

constants $h,\,l$ with double subscripts are determined directly through mechanical steels of transversely isotropic material.

Taking into account the expressions for the component movements of $w_k(r)$ and $u_k(r)$, we obtain the dependences for $t_{ri}(r)$ (5.16) and further for $\sigma_{rz}(r,z)$ according to (5.15).

Stresses $\sigma_z(r,z)$ are determined according to (4.2) as follows:

$$\sigma_z(r,z) = \sum_{i=1,3}^{5} P_i(2z/h)s_{zi}(r) . \tag{5.17}$$

The components of $s_{zi}(r)$ are according to (4.3):

$$s_{zi}(r) = p_{i3}w_3 + g_{i1}\phi_1 + g_{i3}\phi_3 + g_{iq}q, \ (i = 1,3,5), \tag{5.18}$$

where

$$p_{13} = -3q_{33}/70;\ g_{11} = -3e_{31}/70; g_{13} = -3e_{33}/70; g_{1q} = -3(1+3e_{3q}/14)/5;$$
$$p_{33} = q_{33}/15;\ g_{31} = e_{31}/15;\ g_{33} = e_{33}/15;\ g_{3q} = (0,5+e_{3q}/3)/5;$$
$$p_{53} = -q_{33}/42;\ g_{51} = -e_{31}/42;\ g_{53} = -e_{33}/42;\ g_{5q} = -e_{3q}/42.$$

Taking into account the dependencies for w_3, ϕ_1, ϕ_3, and $q(r)$, the components of $s_{zi}(r)$ are determined according to (5.18), and the stresses of $\sigma_z(r,z)$ are determined by formulas (5.17).

Stresses $\sigma_\theta(r,z)$ are determined as follows:

$$\sigma_\theta(r,z) = \sum_{i=1,3}^{5} P_i(2z/h)s_{\theta i}(r) , \tag{5.19}$$

where

$$s_{\theta i}(r) = b_{\theta i1}\frac{u_1}{r} + b_{\theta i2}\frac{u_3}{r} + c_{\theta i1}u_1' + c_{\theta i2}u_3' + b_{\theta i3}w_3 + b_{\theta iq}q,\ (i = 1,3,5) , \tag{5.20}$$

$$b_{\theta 11} = d_0 - 3e_{31}d_{10}/70;\ b_{\theta 12} = -3e_{33}d_{10}/70;\ c_{\theta 11} = d_0\nu - 3e_{31}d_{10}/70;\ c_{\theta 12} = b_{\theta 12};$$
$$b_{\theta 13} = -3q_{33}d_{10}/70;\ b_{\theta 1q} = -3d_{10}(1+e_{3q}/14)/5;$$

$$b_{\theta 31} = c_{\theta 31} = e_{31}d_{10}/15;\ b_{\theta 32} = d_0 + e_{33}d_{10}/15;\ c_{\theta 32} = d_0\nu + e_{33}d_{10}/15;$$
$$b_{\theta 33} = q_{33}d_{10}/15;\ b_{\theta 3q} = d_{10}(1/10 + e_{3q}/15);$$

$$b_{\theta 51} = c_{\theta 51} = -e_{31}d_{10}/42;\ b_{\theta 52} = c_{\theta 52} = -e_{33}d_{10}/42;$$
$$b_{\theta 53} = -q_{33}d_{10}/42;\ b_{\theta 5q} = -e_{3q}d_{10}/42.$$

Taking into account the expressions for $u_1(r), u_3(r), w_3(r)$, $q(r)$ the components $s_{\theta i}(r)$ are obtained according to (5.20), and the stresses $\sigma_\theta(r,z)$ are determined from (5.19).

On the basis of the dependencies (5.1)–(5.20) obtained above for the displacement and stress components, it is possible to set and solve in a new formulation the boundary value problems for circular and annular transtropical plates of arbitrary constant thickness under the action of axisymmetric loads uniformly distributed skew-symmetrically over the annular region at various boundary conditions on the side surface. Solutions of boundary value problems under the action of loads applied uniformly along the line of a circle and in a circular area are obtained using the boundary transition.

6. Boundary conditions

6.1. Boundary conditions from the Reissner equation. Boundary conditions for axisymmetric deformation of round and ring plates in polar coordinates in the K0-N approximation are obtained from (2.1) and have the form:

$$\int_{(s)} \{ \sum_{j=0,1}^{N} \frac{h}{2j+1} (s_{rj} - r_{sj}) \delta u_j + \sum_{j=0,1}^{N-1} \frac{h}{2j+1} (t_{rj} - z_{sj}) \delta w_{j+1} \} ds = 0. \quad (6.1)$$

In (6.1) r_{sj}, z_{sj} are the components of the intensity component of the external axisymmetric load $R_\nu(z)$, $Z_\nu(z)$, which are projections on the axes of coordinates Or and Oz (radial and transverse directions):

$$R_\nu(z) = \sum_{j=1,2}^{N} P_j(2z/h) r_{sj},\ Z_\nu(z) = \sum_{j=0,1}^{N-1} P_j(2z/h) z_{sj},\ (r_{sj},\ z_{sj} - const).$$

Components r_{sj}, z_{sj} are constant coefficients and are determined by the intensity of external axisymmetric load $R_\nu(z)$, $Z_\nu(z)$ as follows:

$$r_{sj} = \frac{2j+1}{h} \int_{-h/2}^{h/2} R_\nu(z) P_j(2z/h) dz,\ (j = 0,1,2,...,N);$$

$$z_{sj} = \frac{2j+1}{h} \int_{-h/2}^{h/2} Z_\nu(z) P_j(2z/h) dz,\ (j = 0,1,...,N-1).$$

For symmetric deformation relative to the median plane in (6.1), terms with even indices are taken in the variations of component displacements, and for obliquely symmetric deformation – with odd ones. In the K13 approximation (obliquely symmetric deformation) in (6.1), terms from δu_1, δu_3, δw_1, δw_3 should be taken.

We will present some boundary conditions later.

6.2. Some boundary conditions for annular plates.

1) Hard clamping of edges $r=b,\ r=a;\ \ b \le r \le a$:

$$u_1(r=b)=0;\ w_1(r=b)=0;\ u_3(r=b)=0;\ w_3(r=b)=0\ ; \tag{6.2}$$

$$u_1(r=a)=0;\ w_1(r=a)=0;\ u_3(r=a)=0;\ w_3(r=a)=0\ .$$

2) Hinged fastening of both edges $r=b,\ r=a;\ \ b \le r \le a$:

$$w_1(b)=0;\ \ s_{r1}(b)=0;\ \ w_3(b)=0;\ \ s_{r3}(b)=0\ ; \tag{6.3}$$

$$w_1(a)=0;\ \ s_{r1}(a)=0;\ \ w_3(a)=0;\ \ s_{r3}(a)=0\ .$$

3) Hard clamping of edge $r=b$ and hinged fastening of edge $r=a;\ \ b \le r \le a$:

$$u_1(r=b)=0;\ w_1(r=b)=0;\ u_3(r=b)=0;\ w_3(r=b)=0\ ; \tag{6.4}$$

$$w_1(a)=0;\ \ s_{r1}(a)=0;\ \ w_3(a)=0;\ \ s_{r3}(a)=0\ .$$

4) Hard clamping of edge $r=a$ and hinged fastening of edge $r=b;\ \ b \le r \le a$:

$$u_1(r=a)=0;\ w_1(r=a)=0;\ u_3(r=a)=0;\ w_3(r=a)=0\ ; \tag{6.5}$$

$$w_1(b)=0;\ \ s_{r1}(b)=0;\ \ w_3(b)=0;\ \ s_{r3}(b)=0\ .$$

5) The outer edge of $r=a$ is rigidly clamped, and the inner edge of $r=b$ is subject to known transverse tangential stresses σ_{rz} and normal stresses σ_r. Then, taking into account the boundary conditions (6.1), we obtain the following boundary conditions in the K13 approximation:

$$t_{r0}(b)=z_{s0}\,;\ t_{r2}(b)=z_{s2}\,;\ s_{r1}(b)=r_{s1}\,;\ s_{r3}(b)=r_{s3}\,; \tag{6.6}$$

$$u_1(r=a)=0;\ w_1(r=a)=0;\ u_3(r=a)=0;\ w_3(r=a)=0\ .$$

6) The outer edge of $r=a$ is tightly clamped, and the inner edge of $r=b$ is subject to known transverse tangential stresses σ_{rz} and, in addition, there are no radial movements of the points of the inner contour of the ring:

$$t_{r0}(b)=z_{s0}\,;\ t_{r2}(b)=z_{s2}\,;\ u_1(b)=0;\ u_3(b)=0; \tag{6.7}$$

$$u_1(r=a)=0;\ w_1(r=a)=0;\ u_3(r=a)=0;\ w_3(r=a)=0\ .$$

7) The outer edge of $r=a$ is tightly clamped, and the inner edge of $r=b$ is free:

$$t_{r0}(b)=0;\; t_{r2}(b)=0;\; s_{r1}(b)=0;\; s_{r3}(b)=0; \tag{6.8}$$

$$u_1(r=a)=0;\; w_1(r=a)=0;\; u_3(r=a)=0;\; w_3(r=a)=0\,.$$

8) The inner edge of $r=b$ is tightly clamped, and the outer edge of $r=a$ is free:

$$u_1(r=b)=0;\; w_1(r=b)=0;\; u_3(r=b)=0;\; w_3(r=b)=0; \tag{6.9}$$

$$t_{r0}(a)=0;\; t_{r2}(a)=0;\; s_{r1}(a)=0;\; s_{r3}(a)=0.$$

9) The inner edge $r=b$ is rigidly clamped, and the outer edge $r=a$ is subjected to transverse tangential stresses σ_{rz} and normal stresses σ_r :

$$u_1(r=b)=0;\; w_1(r=b)=0;\; u_3(r=b)=0;\; w_3(r=b)=0; \tag{6.10}$$

$$t_{r0}(a)=z_{s0};\; t_{r2}(a)=z_{s2};\; s_{r1}(a)=r_{s1};\; s_{r3}(a)=r_{s3}\,.$$

10) The inner edge $r=b$ is tightly clamped, and the outer edge $r=a$ is subject to the known transverse tangential stresses σ_{rz} and, in addition, there are no radial displacements of the points:

$$u_1(r=b)=0;\; w_1(r=b)=0;\; u_3(r=b)=0;\; w_3(r=b)=0; \tag{6.11}$$

$$t_{r0}(a)=z_{s0};\; t_{r2}(a)=z_{s2};\; u_1(a)=0;\; u_3(a)=0.$$

So, in all cases of boundary conditions (6.2)–(6.11) for the ring plate, we have eight linear algebraic equations, from which eight constant integrations are determined: $A_0, B_0, C_0, D_0, A_1, B_1, A_2, B_2$, which are included in the displacement and stress components.

6.3. Some boundary conditions for circular plates. Definition of constants integration. From the boundedness of the functions at $r\to 0$, we obtain from (5.2) for $w_3(r)$ that $D_0=0$. From the expression (5.2) for $w_1(r)$ we get $C_0=0$. It follows from the expressions (5.3) for $A_{11}(r\sqrt{s_1}),\ A_{22}(r\sqrt{s_2})$ that $B_1=0,\ B_2=0$.

Therefore, from the limitation of the components $w_1(r)$ and $w_3(r)$ at $r\to 0$, constants of integration $C_0=0, D_0=0, B_1=0, B_2=0$ were obtained. All displacement components and stress components expressed by formulas (5.13)–(5.18) will also have finite values at $r\to 0$.

We note the following. The components $s_{\theta i}(r)$ according to (5.20) in the stresses $\sigma_\theta(r,z)$ (formula (5.19)) in the first two terms contain r in the denominator. But, taking into account the expressions (5.10) for $u_1(r)$ and $u_3(r)$, these terms at $r\to 0$ also take finite values.

The constants of integration A_0, B_0, A_1, A_2 are determined from the four boundary conditions at the edge of plate $r = a$.

Let's give the boundary conditions for rigid and hinged fixing of the edge.

For the rigidly clamped edge $r = a$ of a circular plate, the boundary conditions have the following form:

$$W(r = a; z) = 0; \;\; U(r = a; z) = 0,$$

or:

$$u_1(a) = 0; \; w_1(a) = 0; \; u_3(a) = 0; \; w_3(a) = 0 . \tag{6.12}$$

For hinged fixing of edge $r = a$:

$$w_1(r = a) = 0; \;\; s_{r1}(r = a) = d_0\,(u_1'(a) + \nu u_1\,(a)/a) = 0 ; \tag{6.13}$$

$$w_3(r = a) = 0; \;\; s_{r3}(r = a) = d_0\,(u_3'(a) + \nu u_1(a)/a) + d_{10}\,\omega_3(a)/15 = 0 .$$

7. Analytical solution of boundary value problems

We present the analytical solution of some axisymmetric boundary value problems for circular and annular transtropical plates of arbitrary constant thickness, which are subjected to skew-symmetric loading in the annular region. We proceed from the analysis of the formulas of general solutions for displacements and stresses (5.1)–(5.19) and boundary conditions (6.2)–(6.13).

1). Boundary problem A. A circular plate of radius a is under the action of a uniform load q_0, which is distributed over the ring area with radii r_1, r_2 $(r_1 \langle\, r_2; \, r_2 \le a)$. Boundary conditions on the side surface can be static, kinematic or mixed.

In all problems A, the components of displacements and stresses at $r = 0$ must be finite. From here we get:

$$C_0 = 0, D_0 = 0, B_1 = 0, B_2 = 0 . \tag{7.1}$$

Task A1. Hinged fastening of edge $r = a$:

$$w_j(r = a) = 0; \;\; s_{rj}(r = a) = 0, \; j = 1, 3 . \tag{7.2}$$

Task A2. Hard clamping of edge $r = a$:

$$w_j(r = a) = 0; \;\; u_j(r = a) = 0, \; j = 1, 3 . \tag{7.3}$$

Task A3. On the side surface of the plate $r = a$, the known intensity of the external load R_ν, Z_ν acts in the projections on the axes of coordinates Or and Oz (radial and transverse directions).

Boundary conditions:

$$s_{rj}(r=a)=r_{sj},\ (j=1,3);\ t_{rj}(r=a)=z_{sj},\ (j=0,2), \tag{7.4}$$

where

$$r_{sj}=\frac{2j+1}{h}\int_{-h/2}^{h/2}R_{\nu}(z,r=a)P_j(2z/h)\,dz;\quad R_{\nu}(z,r=a)=\sum_{j=1,3}^{3}P_j(2z/h)r_{sj};$$

$$z_{sj}=\frac{2j+1}{h}\int_{-h/2}^{h/2}Z_{\nu}(z,r=a)P_j(2z/h)\,dz;\ Z_{\nu}(z,r=a)=\sum_{j=0,2}^{2}P_j(2z/h)z_{sj}.$$

At the same time, the transverse load on the side surface of the plate must balance the external transverse load on the front faces.

Task A4. A known external load of intensity R_{ν} acts on the side surface of plate $r=a$, which is stationary in the transverse direction.

Boundary conditions:

$$w_j(r=a)=0;\ s_{rj}(r=a)=r_{sj},\ (j=1,3). \tag{7.5}$$

Task A5. A known external load of intensity Z_{ν} acts on the lateral surface of plate $r=a$, stationary in the radial direction, which balances the external transverse load acting on the front faces.

Boundary conditions:

$$u_j(r=a)=0,\ (j=1,3);\ t_{rj}(r=a)=z_{sj},\ (j=0,2). \tag{7.6}$$

The constants A_0, B_0, A_1, A_2 are found uniquely from the boundary conditions (7.2). Analytical solutions of these problems – SSS components are expressed by formulas (5.1), (5.6), (5.13), (5.15), (5.17), (5.19) taking into account (5.2)–(5.5), (5.7)–(5.12)), (5.14), (5.16), (5.18), (5.20). In the formulas for the displacement and stress components, the constants C_0, D_0, B_1, B_2 are equal to zero, and the constants A_0, B_0, A_1, A_2 are determined from the boundary conditions (7.2)–(7.6).

2). Boundary problem B. An annular plate with radii b, a $(b \langle\ a)$ is under the action of a uniform load q_0, which is distributed over the annular area. In this case, we will have eight boundary conditions – four boundary conditions each (at the inner and outer edges). From them, all eight unknown constants will be found. In the future, the analytical solutions for the SSS components of the boundary value problems are determined by the corresponding formulas of point 5.

8. Conclusions

A methodology for solving inhomogeneous systems of high-order differential equations of equilibrium for plates of arbitrary constant thickness has been developed in a general form, which consists in reducing them to homogeneous and inhomogeneous differential equations of the second order.

The general solutions of the deterministic systems of high-order differential equations are obtained, followed by the determination of the general solutions of the initial systems of differential equilibrium equations.

Analytically solved axisymmetric boundary value problems for circular and annular transtropical plates under the action of distributed transverse loads in the annular region: obtained general solutions for displacements and stresses; obtained analytical solutions for various boundary conditions.

The resulting solutions can be used to find the SSS of the specified plates when they are loaded along the circle line and in the area of the circle.

This method: 1) makes it possible to significantly simplify the finding of partial and general solutions of initial systems of differential equilibrium equations, especially for loads that are not continuous; 2) significantly changes the methodology of applying the methods of mathematical physics, in particular, the methods of integral transformations, which can be applied not to the initial systems of equations of high orders, but to the obtained inhomogeneous differential equations of the second order.

There are no fundamental difficulties in deriving general solutions for other approximations of the variant of the mathematical theory.

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