Lyudmyla Malyarets, Doctor of Economic Sciences, Professor,
Head of the Department
Simon Kuznets Kharkiv National University of Economics
Kharkiv, Ukraine

Anatoly Voronin, Candidate of Technical Sciences, Associate Professor
Simon Kuznets Kharkiv National University of Economics
Kharkiv, Ukraine

Stepan Lebediev, Senior Lecturer
Simon Kuznets Kharkiv National University of Economics
Kharkiv, Ukraine

DOI: https://doi.org/10.30525/978-9934-26-398-9-13

MATHEMATICAL MODEL OF THE DYNAMICS OF THE INTERACTION OF GOODS PRICES ON ADJACENT MARKETS: THE PROBLEM OF LIMIT CYCLES

Although the problem of the formation of market prices, the determination of equilibrium prices and their stability is well known, and many theoretical works and works summarizing the results of observations are devoted to its research, it is still relevant. This is especially useful for forecasting the dynamics of equilibrium price stability processes relative to changes in the parameters characterizing the state of the system, as this requires the use of fairly complex mathematical instrumentation, namely differential and integral calculus. The purpose of this work is to build a mathematical model of economic dynamics, which would allow, in general, to carry out a qualitative analysis (according to phase trajectories) of the processes that determine the state of equilibrium with respect to prices on adjacent commodity markets. We will consider a general example of a dynamic system that describes the interaction of commodity prices on two markets, and we will show that in such a system there can be six limit cycles that determine the possible variants of its stationary state.

The mathematical model of economic dynamics for the case of two commodity markets can be represented as a system of two ordinary differential equations:

$$\begin{cases} p_1' = F_1(p_1,\, p_2); \\ p_2' = F_2(p_1,\, p_2), \end{cases} \tag{1}$$

where $p_1(t)$ and $p_2(t)$ are changing over time the prices of goods in the corresponding markets (hereinafter, differentiation is carried out by time);

$F_1(p_1, p_2)$, $F_2(p_1, p_2)$ are functions of excess demand for a product in each of the markets, which are nonlinear of degree no higher than two.

It should be noted that for a quadratic system of two ordinary differential equations, the number of limit cycles at a singular point of the “complex focus” type cannot be more than three [1, 2 etc.]. There is a well-known example that shows that for a quadratic system in which there are two foci, the number of limit cycles can reach six in a ratio of 3:3, i.e., three cycles around each focus [3].

To analyze system (1), we choose the following type of excess demand functions:

$$F_1(p_1, p_2) = \lambda p_1 - p_2 + kp_1^2 + mp_1 p_2 + np_2^2,$$

$$F_2(p_1, p_2) = p_1 + ap_1^2 + bp_1 p_2,$$

where a, b, λ, k, m, n are some constant parameters.

In this case, the system has the form:

$$\begin{cases} p_1' = \lambda p_1 - p_2 + kp_1^2 + mp_1 p_2 + np_2^2; \\ p_2' = p_1 + ap_1^2 + bp_1 p_2. \end{cases} \quad (2)$$

System (2) has two equilibrium positions. These are points $p_1^* = 0;\ p_2^* = 0$ and $p_1^* = 0;\ p_2^* = 1/n$, which, for certain parameter values, are foci.

Let us transform system (2) by introducing new variables such as $\widehat{p}_1 = p_1$ and $\widehat{p}_2 = p_2 - p_2^*$, where the variable p_2^* can take the value either 0 or $1/n$. Then system (2) takes the form:

$$\begin{cases} \widehat{p_1'} = (\lambda + mp_2^*)\widehat{p}_1 + (2np_2^* - 1)\widehat{p}_2 + k\widehat{p}_1^2 + m\widehat{p}_1\widehat{p}_2 + n\widehat{p}_2^2; \\ \widehat{p_2'} = (1 + bp_2^*)\widehat{p}_1 + a\widehat{p}_1^2 + b\widehat{p}_1\widehat{p}_2. \end{cases} \quad (3)$$

It is obvious that for values of the parameter λ close to $\lambda_1 = 0$ or to $\lambda_2 = -m/n$ provided that $\omega^2 = (1 + bp_2^*)(1 - 2np_2^*) > 0$, there are two foci.

Assuming that $b = -2n$, then $\omega^2 = (1 - 2np_2^*)^2$ and for $p_2^* \in \{0; 1/n\}$ we obtain $\omega^2 = 1$. Thus, oscillations with a frequency of $\omega = 1$.

If we take $\widehat{p}_1 = x_1$, $\widehat{p}_2 = x_2$, $\lambda = \lambda_1 = 0$ for the equilibrium position $p_1^* = 0;\ p_2^* = 0$ (first focus), then we obtain system (3) in normal form:

$$\begin{cases} x_1' = -x_2 + kx_1^2 + mx_1 x_2 + nx_2^2; \\ x_2' = x_1 + ax_1^2 - 2nx_1 x_2. \end{cases} \quad (4)$$

If we accept $\widehat{p}_1 = x_1$, $\widehat{p}_2 = -x_2$, $\lambda = \lambda_2 = -m/n$ for the equilibrium position $p_1^* = 0$; $p_2^* = 1/n$ (second focus), then we obtain system (3) in normal form:

$$\begin{cases} x_1' = -x_2 + kx_1^2 - mx_1x_2 + nx_2^2; \\ x_2' = x_1 - ax_1^2 - 2nx_1x_2. \end{cases} \tag{5}$$

Obviously, systems (4) and (5) can be written in the general form, using the notation $a_0 = \pm a$, $m_0 = \pm m$, where the “+” sign corresponds to system (4), and the “–” sign corresponds to system (5):

$$\begin{cases} x_1' = -x_2 + kx_1^2 + m_0x_1x_2 + nx_2^2; \\ x_2' = x_1 + a_0x_1^2 - 2nx_1x_2. \end{cases} \tag{6}$$

It is convenient to move from the system of two differential equations (6) to one complex-valued differential equation by making a change of variables $z = x_1 + i \cdot x_2$ and $\overline{z} = x_1 - i \cdot x_2$, where $i^2 = 1$. We get:

$$z' = i \cdot z + 0,5g_{20}z^2 + g_{11}z \cdot \overline{z} + 0,5g_{02}\overline{z}^2, \tag{7}$$

where the parameters of the equation are $g_{20} = 0,5\left(k - 3n + i(a_0 - m_0)\right)$, $g_{11} = 0,5(k + n + i \cdot a_0)$, $g_{02} = 0,5\left(k + n + i(a_0 + m_0)\right)$.

To study the multiplicity of limit cycles, it is necessary to calculate the values of the first three Lyapunov quantities. If they are all equal to zero, then system (6) will be conservative with an infinite number of periodic trajectories. Formulas for calculating Lyapunov quantities were proposed in [4]:

$$\begin{aligned} & l_1 = -\frac{1}{2}\operatorname{Im}\left(g_{20}g_{11}\right); \\ & l_1 = 0, \quad l_2 = -\frac{1}{12}\operatorname{Im}\left((g_{20} - 4\overline{g}_{11})(g_{20} + \overline{g}_{11})\overline{g}_{11}g_{02}\right); \\ & l_1 = l_2 = 0, \quad l_3 = -\frac{5}{64}\left(4\left|g_{11}\right|^2 - \left|g_{02}\right|^2\right) \cdot \operatorname{Im}\left((g_{20} + \overline{g}_{11})\overline{g}_{11}^2 g_{02}\right). \end{aligned} \tag{8}$$

From relations (8) it follows that the cyclicity of each focus can be determined using the following expression:

$$g_{20} = r \cdot \overline{g}_{11}. \tag{9}$$

Let’s analyze the possible states of the system using the relation (9).

When $r = -1$ there is a conservative case with an infinite set of periodic regimes.

If r is a complex number, then there is a unique limit cycle.

If r is a real number and $r \neq -1$ or $r \neq 4$, then there are two limit cycles.

If $r = 4$, then there are three limit cycles.

Calculate the values of the first three Lyapunov quantities based on relationships (8). Equalities $l_1 = l_2 = 0$ take place if the following relations are satisfied:

$$3k + 7n = 0 \tag{10}$$

$$m_0 - 5a_0 = 0 \tag{11}$$

Note that from (11) it follows that for each equilibrium positions it is true that $m = 5a$.

Let's assume that $k = -\frac{7}{3}n$ and $m = 5a$. Then system (6) has the form:

$$\begin{cases} x_1' = -x_2 - \frac{7}{3}nx_1^2 + 5a_0x_1x_2 + nx_2^2; \\ x_2' = x_1 + a_0x_1^2 - 2nx_1x_2. \end{cases} \tag{12}$$

For these parameter values, the third Lyapunov quantity is determined by the relation:

$$l_3 = \frac{25na_0}{1728}(n^2 - 6a_0^2)(16n^2 - 51a_0^2). \tag{13}$$

Obviously, if the parameters and are rational numbers and are not equal to zero, then, in accordance with relation (13), the third Lyapunov quantity is nonzero: $l_3 \neq 0$. This means that for the same values of the parameters of the nonlinear part of systems of equations (4) and (5), there are three limit cycles around each equilibrium position. Thus, there are simultaneously six limit cycles in this system in a ratio of 3:3.

Therefore, the analytical model of economic dynamics which we present in this paper allows to study the dynamics of prices in adjacent markets depending on the value of the model parameters and, accordingly, can be the basis for simulation modeling. When applied to the model of two markets, this study demonstrates the possibilities of qualitative forecasting (namely, according to phase trajectories) of periodic regimes of market conditions. Application of the system of differential equations in this mathematical model of economic dynamics allows considering processes that are continuous in time.

References:

1. Bamor R. (1986) Quadratic vector fields in the plane have a finite number of limit cycles. *Publications mathématiques de l'I.H.É.S.* Vol. 64, pp. 111–142.

2. Chicone C., Shafer D. S. (1983) Separatrix and limit cycles of quadratic systems and Dulac's theorem. *Transactions of the American Mathematical Society*. Vol. 278, no. 2, pp. 585–612.

3. Voronin A. V., Lebedev S. S. (2023) Quadratic system of two differential equations with six limit cycles: two approaches to problem analysis. *ResearchGate*. February 16. DOI: https://doi.org/10.13140/rg.2.2.32572.92808

4. Zoladek H. (1994) Quadratic systems with center and their perturbations. *J. Differential Equations*. Vol. 109, no. 2, pp. 223–273.