CHAPTER «PHYSICAL AND MATHEMATICAL SCIENCES»

THE METHOD OF INTEGRATING SYSTEMS OF HIGH-ORDER EQUI-LIBRIUM EQUATIONS OF THE MATHEMATICAL THEORY OF THICK PLATES UNDER INTERMITTENT LOADS (PART 1)

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Abstract. A method for integrating inhomogeneous systems of differential equilibrium equations for transversely isotropic plates of arbitrary thickness under the action of various transverse loads, which can be intermittent or concentrated, is proposed. These systems are of high order. Such systems of inhomogeneous equations are obtained on the basis of a developed variant of mathematical theory, which is built on the following provisions: 1) the components of the stress-strain state are considered functions of three coordinates; 2) the components of the displacements are represented in the form of infinite mathematical Fourier-Legendre series in the transverse coordinate; 3) the remaining components of the stress-strain state (SSS) using the three-dimensional equilibrium equations of the theory of elasticity and the Reissner variational principle are also depicted in the form of infinite mathematical series using Legendre polynomials; 4) differential equations of equilibrium in displacements and boundary conditions on the lateral surface are derived from the variational Reissner equation. It should be noted that the boundary conditions on the flat faces of the plates are satisfied exactly. Boundary problems based on this

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variant of mathematical theory are solved taking into account a certain number of members in the mathematical series for SSS components. As the number of terms increases, the order of the systems of differential equations increases, but the accuracy also increases. The direct analytical solution of systems of high-order equilibrium equations is associated with rather great mathematical difficulties, especially when finding partial solutions. If the load that acts on the plate is intermittent or local, then additional difficulties arise. The subject of the study is the inhomogeneous systems of high-order partial differential equations for various right-hand sides and a method for solving them. The purpose of the work is to build an effective method for solving such systems in order to find their partial and general solutions. The research methodology and conclusions are as follows. 1). By means of algebraic and differential transformations, systems of high-order equilibrium equations are reduced to convenient defining systems of inhomogeneous differential equations of the same order with respect to new unknown functions. 2). Each inhomogeneous differential equation of higher order is reduced by the method of decreasing the order to inhomogeneous differential equations of the second and fourth orders. 3). Partial solutions of higher-order inhomogeneous equations of defining systems are represented as a differential operator from a linear combination of partial solutions of second-order and fourth-order inhomogeneous equations. 4). In the future, it is recommended to apply mathematical methods of solution to equations of low order, including methods of integral transformations.

5) The general solution of the defining inhomogeneous system of high-order differential equations is presented through the general and partial solutions of the second-order and fourth-order inhomogeneous equations. 6) General solutions of the original systems of differential equations of high-order equilibrium are found by inverse transformations through general and partial solutions of inhomogeneous equations of the second and fourth orders. The forms of general solutions of inhomogeneous systems of high-order differential equations of equilibrium in displacements are obtained. Using the proposed method, a general solution to the inhomogeneous system of eighth-order governing equations of axisymmetric problems of bending transversely isotropic circular and annular plates of arbitrary thickness under intermittent and local loads, which are expressed through the Dirac delta function, is obtained. To find partial solutions, the Hankel integral transform was used. Proposed method significantly simplifies the finding of partial

and general solutions of inhomogeneous systems of high-order differential equations, especially in cases where the right-hand sides of the equations are discontinuous or local functions that correspond to discontinuous or local loads. The developed technique for solving boundary-value problems can also be applied to plate theories that use physical and geometric hypotheses.

1. Introduction

It is known that classical theories [27, p. 3; 32, p. 22; 33, p. 84; 36, p. 51; 37, p. 69; 38, p. 64; 39, p. 99; 52, p. 569; 53, p. 84; 54, p. 63; 55, p. 67; 56, p. 127; 57, p. 54; 60, p. 21] unsatisfactorily describe the stress-strain state (SSS) of plates and shells under the action of local, discontinuous, rapidly changing loads, in the presence of holes and sharp changes in mechanical parameters, with a significant thickness of the elements, and in other cases that lead to significant changes in the SSS. Tymoshenko-Reisner type theories [41, p. 184; 42, p. 495; 47, p. 744], in particular [2, p. 1004; 4, p. 486; 11, p. 675; 31, p. 993; 34, p. 195] satisfactorily describe SSS of thin plates, if the external load is smooth, and the boundary conditions do not lead to the appearance of marginal potential effects. Refined theories of plates and shells [3; 5, p. 239; 6, p. 107; 10, p. 49; 25, p. 663; 26, p. 669; 40], based on various physical and geometric hypotheses, do not describe with sufficient accuracy the SSS for a wide class of boundary value problems, since qualitative changes in all components of displacements and stresses in thickness cannot be depicted objectively in principle. In these theories, boundary effects are not always fully taken into account. The resulting systems of nonhomogeneous partial differential equations allow for Not all edge effects and, as a rule, are of low order. It should be noted that the accuracy of the the-ories depends on how accepted the hypotheses adequately reflect the qualitative nature of the change in the SSS in the thickness and whether all the components of the SSS are taken into account. It should be noted that from this point of view, a model of the image of SSS in plates based on the six-parameter theory deserves attention [1, p. 242].

Studies [7; 8; 24, p. 238; 48] have important theoretical and applied value for studying boundary value problems using refined theories.

Analytical solution of boundary value problems for plates and shells based on the three-dimensional theory of elasticity [20; 28], which essentially should give the most accurate solution, is associated with significant and often insurmountable mathematical difficulties. Only in some special cases [13;

30, p. 49] we can obtain analytically accurate solutions of three-dimensional differential equations (DE). Hence, the scientific problem arises: on the one hand, solutions of applied boundary value problems for plates and shells with a sufficiently high accuracy are required, and on the other hand, achieving this goal involves great difficulties. The solution to this problem is an actual area of research in the mechanics of plates and shells of arbitrary thickness.

The purpose of solving this global problem in the mechanics of deformed solids is to construct new variants of the mathematical theory of plates and shells of arbitrary thickness, which would take into account all components of the SSS with high accuracy; developing effective mathematical methods for solving boundary-value problems in accordance with these theories; construction of mathematical methods for solving nonuniform DE in high-order partial derivatives that arise in these boundary-value problems.

The first versions of the theory of plates and shells, in which the expansion of SSS components into infinite series along the transverse coordinate was used, were proposed, in particular, in studies [9, p. 238; 21; 35, p. 335; 50, p. 191]. Later, this mathematical approach was developed in publications [14, p. 49; 15; 17, p. 83; 18; 19, p. 77; 33, p. 84; 36, p. 51; 37, p. 69; 38, p. 64; 39, p. 99; 44, p.21; 52, p. 569; 53, p. 84; 54, p. 63; 55, p. 67; 56, p. 127; 57, p. 54; 62, p. 51; 63, p.164] in which the Legendre polynomials were used.

An overview of the research of models for calculating the SSS of plates and shells is presented, in particular, in publications [7; 18; 27; 32, p. 22].

Based on the Prusakov approach [36, p. 51], variants of the mathematical theory of plates and shallow shells of arbitrary thickness under the action of arbitrary transverse loads using interrelated equations were developed [52, p. 569; 53, p. 84; 54, p. 63; 55, p. 67; 56, p. 127; 57, p. 54; 58, p. 154; 59, p. 60; 60, p. 21; 61, p. 496; 62, p. 51]. In many other works of the author, new variants of the mathematical theory of single-layer and multilayer, linear and nonlinear elastic plates and shells of arbitrary thickness were constructed. The main dependences and systems of DE in different approximations were obtained. Methods for their solution were developed, and numerical results were obtained. In the future, we will apply our variant of mathematical theory (MT). The method of obtaining the basic equations includes the following steps: 1) representation of components of displacements, strains and stresses in the form of infinite mathematical series along the transverse coordinate using Legendre polynomials; 2) integration of three-dimensional differential equilibrium equations; 3) the use of the Reissner variational principle [43, p. 90]; 4) obtaining interrelated equations, which in the decompositions of the components of the displacement simultaneously take into account a certain number of members of the serie. It should be noted that the components of the SSS in the considered variant of MT are functions of three coordinates. Taking into account the interconnected equations, in contrast to the energy-asymptotic method [33, p. 84; 39, p. 99], we can immediately solve the boundary-value problem using several members of each mathematical series. This increases the order of the system of DE, but also increases the accuracy of the solution [57, p. 54].

We also emphasize that in our variant of the MT, the boundary conditions on the upper and lower faces of the plates (upper and lower boundary surfaces of the shells) are exactly satisfied, unlike many other theories. In combination with the use of the Reissner variational principle, additional studies have shown that this increases the efficiency of SSS determination in comparison with other approaches [57, p. 54; 60, p. 21], especially for thick plates and shells.

It is a priori clear that the accuracy of any theory is determined by the number of unknown independent functions of the three coordinates that characterize the SSS of the plate and the shell of arbitrary thickness, subject to adequate assumptions about the change of all components of the SSS in the plate (shell). These functions can be called theory parameters, or degrees of freedom of theory. With an increase in the number of sought functions, the accuracy of solving boundary value problems increases. However, at the same time, the order of the systems of differential equilibrium equations and the mathematical complexity of solving these systems also increase.

A variant of the MT makes it possible to determine with any accuracy all the components of the SSS, which include the components of the internal SSS and the components of the SSS determined by vortex and potential boundary effects. The internal stress-strain state is caused by normal and tangential stresses that are applied to the lateral surface and do not self-balance in thickness. Vortex and potential boundary effects arise from the action of normal and tangential stresses, which are applied to the lateral surface and self-equilibrium in thickness. The SSS, which depends on the vortex boundary effects, is caused by tangential stresses parallel to the median plane and self-balanced in thickness. The SSS, which depends on the potential boundary effects, is caused by normal and transverse tangent stresses, self-balanced in thickness.

Timoshenko-Reissner type theory takes into account the vortex boundary effect and the internal SSS in the first approximation and does not account for the potential boundary effect.

A variant of MT contains in infinite mathematical series for the components of tangential displacements $U(x, y, z)$, $V(x, y, z)$ (formulas (2.3a)) members of the form $P_k(2z / h) u_k(x, y)$, $P_k(2z / h) v_k(x, y)$, where $k = 0, 1, \dots$. Infinite mathematical series for transverse displacements of $W(x, y, z)$ (formulas (2.3b)) contain members of the form $P_k(2z / h) w_k(x, y)$, where $k = 1, 2, \dots$. If in tangential displacements (2.3) we take into account additives with indices 0, 1, 2,…, N, where n is an odd integer, then we call this approximation the approximation K0-N; if we consider only terms with indexes 1, 3, then we will call this approximation the approximation K13.

In approximation K1, three unknown functions are taken into account in the mathematical seriess (2.3a), (2.3b): $u_1(x, y), v_1(x, y), w_1(x, y)$. The system of differential equilibrium equations is of the sixth order. The approximation of K01 takes into account five unknown functions: $u_0(x, y)$, $u_1(x, y), v_0(x, y), v_1(x, y), w_1(x, y)$. The system of DE of equilibrium has the tenth order.

Six functions are taken into account in the K13 approximation: $u_1(x, y)$, $u_3(x, y), v_1(x, y), v_3(x, y), w_1(x, y)$, $w_3(x, y)$. The system of DE of equilibrium has a twelfth order. In the K0-3 approximation, eleven functions are taken into account: $u_0(x, y)$, $u_1(x, y), \dots$, $u_3(x, y)$, $v_0(x, y)$, $v_1(x, y), \dots, v_3(x, y)$, $w_1(x, y)$, $w_2(x, y)$, $w_3(x, y)$. The system of DE of equilibrium has a twenty-second order.

In the K135 approximation, nine functions are taken into account: $u_1(x, y)$, $u_3(x, y)$, $u_5(x, y)$, $v_1(x, y)$, $v_3(x, y), v_5(x, y)$, $w_1(x, y)$, $w_3(x, y)$, $w_5(x, y)$. The system of DE of equilibrium has an eighteenth order. Seventeen functions are taken into account in the approximation of K0-5: $u_0(x, y), u_1(x, y), \dots, u_5(x, y)$, $v_0(x, y), v_1(x, y), \dots, v_5(x, y), w_1(x, y), \dots, w_5(x, y)$. The system of DE of equi-librium has the thirty-fourth order.

Currently, in the publications of Western publications, research is usually carried out on the basis of theories such as Timoshenko-Reissner or their modifications [2, p. 1004; 5, p. 239; 6, p. 107; 10, p. 49; 11, p. 675; 31, p. 993; 34, p.195]. These theories essentially correspond to the partial case of our variant of theorie namely, the K01 approximation. The solution of boundary value problems in the second and higher approximations leads to the necessity of finding particular and general solutions of systems of nonhomogeneous partial differential equations of high order. This is due to fairly large mathematical difficulties. In our opinion, this is one of the reasons that prevents many researchers from analytically developing the direction in the mechanics of plates and shells, associated with the construction of new variants of MT, which makes it possible to determine the SSS of plates and shells of arbitrary thickness with high accuracy.

Advantages of the mathematical theory variant are the ability to solve boundary value problems for plates and shells with any high accuracy. As the number of additives taken into account in these series increases, the accuracy of finding the SSS components in the plates (shells) increases. This increases the accuracy of determining the components of the internal SSS and components of eddy and potential boundary effects.

The novelty of the work is to construct an effective method for solving inhomogeneous systems of high order equilibrium differential equations, which take place in the boundary value problems of a variant of mathematical theory of arbitrary thickness plates.

2. Physical and mathematical formulation of boundary problems for plates of arbitrary thickness

2.1. Physical statement of boundary problems. From the standpoint of the three-dimensional theory of elasticity, we consider a transtropic plate of arbitrary constant thickness h in a rectangular coordinate system x, y, z. The axes x, y lie in the median plane, the z axis is perpendicular to it and directed upwards $-h/2 \le z \le h/2$). On the horizontal faces of the plate are applied static transverse loads $q_1(x, y)$ and $q_2(x, y)$, which are directed downwards. All SSS components are functions of three coordinates. Boundary conditions on the face planes of the plate:

$$\sigma_z(z = h/2) = -q_1(x, y)\ ;\ \sigma_z(z = -h/2) = q_2(x, y)\ ; \tag{2.1a}$$

$$\sigma_{xz}(z = \pm h/2) = \sigma_{yz}(z = \pm h/2) = 0\ . \tag{2.1b}$$

The transverse load on the horizontal faces is represented by the sum of the skew symmetric $q/2$ and symmetric $p/2$ loads relative to the median plane. Then:

$$\sigma_z(z = \pm h/2) = (\mp q(x, y) - p(x, y))/2\ ,\ p(x, y) = q_1(x, y) - q_2(x, y)\ ,$$

$$q(x, y) = q_1(x, y) + q_2(x, y)\ .$$

The boundary conditions on the face planes of the plate under oblique symmetry loading relative to the median plane according to (2.1a) and (2.1b) shall be written as follows:

$$\sigma_z(z=\pm h/2)=\mp q(x,y)/2\,,\ \sigma_{xz}(z=\pm h/2)=\sigma_{yz}(z=\pm h/2)=0\,. \quad (2.2)$$

The conditions on the side surface of the plate may be static, kinematic or mixed. The stresses, deformations and displacements in the plate are unknown functions of the three coordinates.

2.2. Mathematical statement of boundary problems. We present the basic mathematical equations and relations that are necessary for solving the boundary value problems of transversal isotropic plates of arbitrary constant thickness.

2.2.1. Displacement in the plate. In the considered variant of the MT of elastic transversely isotropic plates of arbitrary thickness, the components of the displacement are presented in the form of Fourier–Legendre series:

$$U(x,y,z)=\sum_{k=0}^{\infty}P_k(2z/h)\,u_k(x,y),\ (U,V;u_k,v_k)\,; \quad (2.3a)$$

$$W(x,y,z)=\sum_{k=1}^{\infty}P_{k-1}(2z/h)w_k(x,y)\,. \quad (2.3b)$$

In dependencies (2.3a), (2.3b) $P_k(2z/h)$–Legendre polynomials; U,V,W–components of displacements (tangential and transverse). Members with odd indices in displacement components correspond to skew symmetric deformation and members with even indices are symmetric.

2.2.2. Components of stresses in the plate. Here are the general structural formulas of the stress components that follow from the DE system of the theory of spatial elasticity and the Reisner variational equation:

$$\sigma_{xz}(x,y,z)=\sum_{i=0}^{\infty}P_i\,t_{xi};\ \sigma_{yz}(x,y,z)=\sum_{i=0}^{\infty}P_i\,t_{yi};\ \sigma_z(x,y,z)=\sum_{i=0}^{\infty}P_i\,s_{zi}; \quad (2.4)$$

$$\sigma_x(x,y,z)=\sum_{i=0}^{\infty}P_i\,s_{xi},\ (\sigma_x\to\sigma_y;s_{xi}\to s_{yi});\ \sigma_{yx}(x,y,z)=\sum_{i=0}^{\infty}P_i\,t_{yxi},$$

where $t_{xi},...,t_{yxi}$–functions that depend on the displacements of $u_k(x,y),v_k(x,y),w_k(x,y)$ and mechanical-geometric parameters (MGP).

2.2.3. Displacement and stresses in the K0-N approximation. The displacement components are determined according to (2.3a), (2.3b):

$$U(x,y,z)=\sum_{k=0}^{N}P_k(2z/h)\,u_k(x,y),\ (U,V;u_k,v_k)\,; \quad (2.\ 5a)$$

$$W(x,y,z)=\sum_{k=1}^{N}P_{k-1}(2z/h)w_k(x,y). \qquad (2.\ 5b)$$

The stress components according to (2.4):

$$\sigma_{xz}(x,y,z)=\sum_{i=0}^{N+1}P_i\,t_{xi};\ \sigma_{yz}(x,y,z)=\sum_{i=0}^{N+1}P_i\,t_{yi}; \qquad (2.6)$$

$$\sigma_z(x,y,z)=\sum_{i=0}^{N+2}P_i\,s_{zi};\ \sigma_x(x,y,z)=\sum_{i=0}^{N+2}P_i\,s_{xi};$$

$$\sigma_y(x,y,z)=\sum_{i=0}^{N+2}P_i\,s_{yi};\ \sigma_{xy}(x,y,z)=\sum_{i=0}^{N}P_i\,t_{yxi}.$$

Transverse normal and shear stresses exactly satisfy conditions (2.1a), (2.1b). For the approximations K0-3 and K0-5 in formulas (2.6), we must take $N=3;\ N=5$, respectively; expressions for these functions are given in [53, p. 84; 56, p. 127].

2.2.4. Differential equilibrium equations. The system of equilibrium equations is inhomogeneous with partial derivatives with respect to the sought components in displacements. The resulting system is divided into two independent. One system describes the SSS of the plate in symmetrical deformation relative to the median plane. Another system describes the SSS at skew symmetry (with purely bending deformation without transverse crimping). In all dependencies and equations, the components of the displacements taken into account in the partial sums of the series (2.5a), (2.5b) must be taken into account. In the approximation K0-N, where N is an odd natural number, the order of the system of differential equations of equilibrium is $(6N+4)$; for symmetric deformation in the approximation K02 (N-1), the system of equations has the order $(3N+1)$, and for the skew-symmetric – order $3(N+1)$.

System of DE of equilibrium for symmetric deformation in the approximation K02 (N-1):

$$\sum_{j=0,2}^{N-1}(M_{iuj}u_j+M_{ivj}v_j)+\sum_{j=2,4}^{N-1}M_{iwj}w_j=M_ip,\quad (i=1,.2,..,(3N+1)/2). \qquad (2.7)$$

System of DE of equilibrium at skew symmetric deformation in the approximation K13 N:

$$\sum_{j=1,3}^{N}(L_{iuj}u_j+L_{ivj}v_j)+\sum_{j=1,3}^{N}L_{iwj}w_j=L_iq,(i=1,.2,..,3(N+1)/2). \qquad (2.8)$$

In the left parts of the systems of equations (2.7), (2.8) L and M with indexes, these are operators of x, y of order not higher than the second. The operators L and M with indices on the right-hand side of these equations are operators of order not higher than the first [53, p. 84; 56, p. 127]. It is shown that the differential matrices of these systems are symmetric.

2.2.5. Boundary conditions on the lateral surface in the approximation K0-N. The boundary conditions are obtained from the Reisner variation equation:

$$\int_{(s)} \Big(\sum_{j=0}^{N} \frac{h}{(2j+1)}((s_{xj}\, l_x + t_{yxj}\, l_y - x_{sj})\,\delta u_j + (t_{yxj}\, l_x + s_{yj} l_y - y_{s\,j})\,\delta v_j) + \qquad (2.9)$$

$$+\sum_{j=0}^{N-1} \frac{h}{(2j+1)}(t_{xj}\, l_x + t_{yj}\, l_y - z_{sj})\delta w_{j+1}\Big)ds = 0 .$$

In (2.9) $l_x,\ l_y$ – is the cosines of the angles between the normal vector to the lateral surface and the coordinate axes; S – contour of the plate; $x_{s\,j}(x,y), y_{s\,j}(x,y),\ z_{s\,j}(x,y)$ – members in mathematical series of the image of the external loading $X_\nu(x,y,z), Y_\nu(x,y,z),\ Z_\nu(x,y,z)$ by Legendre polynomials:

$$x_{sj}(x,y) = (2j+1)\Big(\int_{-h/2}^{h/2} X_\nu(x,y,z)P_j(2z/h)dz\Big)/h, \qquad (2.10)$$

$$(x_{sj} \to y_{sj}; X_\nu \to Y_\nu;\ j = 0,1,...,N);$$

$$z_{s\,j}(x,y) = (2j+1)\Big(\int_{-h/2}^{h/2} X_\nu(x,y,z)P_j(2z/h)dz\Big)/h,\ (j = 0,1,...,N-1),$$

where Z_n must balance the transverse load on the upper and lower surfaces of the plate.

Equations (2.9) yield different boundary conditions. Here are some of them.

1). Boundary conditions in displacements. Only the displacement compo-nents $U_\Gamma(x,y,z),\ V_\Gamma(x,y,z),\ W_\Gamma(x,y,z)$ are known on the side surface Γ of the plate. Boundary conditions:

$$u_j(x,y) = u_{j\Gamma}(x,y);\ v_j(x,y) = v_{j\Gamma}(x,y),\ (j = 0,1,...,N); \qquad (2.11)$$

$$w_j(x,y) = w_{j\Gamma}(x,y),\ (j = 1,...,N);\quad x,y \in S,$$

where

$$u_{j\Gamma}(x,y)=\frac{2j+1}{h}\int_z U_\Gamma(x,y,z)\,P_j(2z/h)\,dz,$$
$$(u_{j\Gamma}\Rightarrow v_{j\Gamma},U_\Gamma\to V_\Gamma),(j=0,1,\dots,N));$$
$$w_{j\Gamma}(x,y)=\frac{2j-1}{h}\int_z W_\Gamma(x,y,z)\,P_{j-1}(2z/h)\,dz,\ (j=1,\dots,N).$$

2). Boundary conditions in stresses. Only the external load $X_\nu(x,y,z)$, $Y_\nu(x,y,z)$, $Z_\nu(x,y,z)$ is specified on the side surface. Then we have the following boundary conditions:

$$s_{xj}(x,y)\,l_x+t_{yxj}(x,y)\,l_y=x_{sj}(x,y); \tag{2.12}$$
$$t_{yxj}(x,y)\,l_x+s_{yj}(x,y)\,l_y=y_{sj}(x,y),\ (j=0,1,\dots,N);$$
$$t_{xj}(x,y)\,l_x+t_{yj}(x,y)\,l_y=z_{sj}(x,y),\ (j=0,1,\dots,N-1);\ x,y\in S.$$

In the approximations K01, K0-3, K0-5, to obtain displacements, stresses and boundary conditions, it is necessary to put $N=1;\ N=3;\ N=5$ in (2.5a), (2.5b), (2.6)–(2.12), respectively.

2.3. Formulation of a general problem in the constructed MT of plates

The problem in the constructed variant of MT is the need to solve systems of inhomogeneous DE in high-order partial derivatives. The main mathematical difficulty is finding partial solutions of high-order inhomogeneous equations. The problem is particularly complicated if the transverse load is discontinuous or local. This problem is also characteristic of other simpler theories

In what follows, we consider the system of inhomogeneous differential equations of equilibrium (2.8) with skew-symmetric deformation in the approximation K13 N.

In [16, p. 21; 23, p. 423; 45, p. 382; 51, p. 147], the method of analytic integration of inhomogeneous systems of DE of equilibrium was that the general solutions of homogeneous systems and partial solutions of inhomogeneous systems were determined directly from the initial systems of equations of equilibrium. In the case of partial solutions, the methods of integral transformations were used. For high-order systems, this technique led to great difficulties.

In this article, a much simpler method for integrating the system of DE of equilibrium is proposed, which is as follows.

1). The initial system of differential equations of equilibrium was reduced to two independent systems of equations. One system (homogeneous) described a vortex boundary effect. Another system (inhomogeneous) described the internal SSS and the potential boundary effect.

2). From the transformed systems of equilibrium equations, two deterministic systems of DE are derived relative to the new sought functions.

3). Deterministic systems of DE were reduced by the method of decreasing order to low-order DE, for which general and partial solutions were sought. Partial solutions of low-order inhomogeneous equations could be found by different methods, in particular, by integral transformations. The general solutions of homogeneous equations of the defining systems were the sum of the total solutions of the corresponding homogeneous equations of low order. Partial solutions of inhomogeneous equations of a high-order definite system were represented as differential operators from a certain linear combination of partial solutions of inhomogeneous low order equations.

4). The general solutions for the components of displacements and stresses were found from the corresponding dependences obtained during the transformation of differential equations.

The described method significantly simplifies the analytical integration of high order equilibrium DE.

3. Justification of the advantages of the developed variant of MT

To demonstrate the efficiency and accuracy of a variant of mathematical plate theory, we present some numerical results of many studies of internal SSS of isotropic and transversal-isotropic plates with different MGP under the action of transverse loading:

$$q(x,y) = q_{mn}\sin(m\pi x/a)\sin(n\pi y/b),$$
$$p(x,y) = p_{mn}\sin(m\pi x/a)\sin(n\pi y/b).$$

The internal SSS is independent of boundary effects. Boundary effects were investigated in [62, p. 51].

Tables 1–4 summarize the results of numerical studies of stresses and transverse displacements for square isotropic and transversal isotropic plates of different thicknesses.

In tables:

$$\tilde{\sigma}_x = \sigma_x(x,y,z)/q(x,y), (\sigma_x, \sigma_z, \sigma_{xz}); \tilde{W} = W(x,y,z)\,E/(q(x,y)\,h).$$

Tables 1, 2 show $\tilde{\sigma}_x$ and $\tilde{W}$ in transversal isotropic plates with different parameters; Δ_{t1}, Δ_{t3}, Δ_{t5} is the percentage difference between the exact solution (ES) and the decision in the appropriate approximation; Δ_{pq} characterizes the effect of transverse crimping in percent.

Table 1

Components of the SSS of a square transropic plate ($h / a = 0,5$; $G' / G = 0,1; E' / E = 1; \nu' = \nu = 0,3;\ m = n = 1; p_{mn} / q_{mn} = 0\ / p_{mn} / q_{mn} = 1$)

$\frac{z}{h}$	**K1** **K01**	**K13** **K0-3**	**K135** **K0-5**	**ES**	Δ_{t5} **%**	Δ_{t3} **%**	Δ_{t1} **%**	$\tilde{\sigma}_x$ **%**
	$\tilde{\sigma}_x$							
0,5	-0,9250 -1,098	-1,587 -1,692	-1,889 -2,016	-1,951 -2,083	3,18 3,22	18,7 18,8	52,6 44,3	6,34 -
0	0 -0,1725	0 -0,0582	0 -0,0620	0 -0,0627	- -	- -	- -	- -
-0,5	0,9250 0,7525	1,587 1,481	1,889 1,761	1,951 1,820	- 3,24	18,6	58,7	7,2 -
	$\tilde{W}$							
0,5	-6,726 -6,726	-6,337 -6,564	-6,277 -6,506	-6,260 -6,486	0,27 0,31	1,23 1,20	7,44 3,70	3,48 -
0	-6,726 -6,726	-6,250 -6,250	-6,225 -6,225	-6,226 -6,226	0,02 -	0,39 -	8,03 -	
-0,5	-6,726 -6,726	-6,337 -6,109	-6,277 -6,049	-6,260 -6,035	- 0,23	1,23	11,4	3,73 -

Tables 3, 4 show $\tilde{\sigma}_x$ and $\tilde{W}$ for the upper face of isotropic square plates of different thicknesses with skew symmetric deformation under smooth and non-smooth loading according to classical theory (CT), exact theory (ET) and approximations K13, K135.

These and our other studies, as well as the results presented in particular in [33, p. 84; 36, p. 51; 37, p. 69; 38, p. 64; 39, p. 99], testify to the high accuracy of the developed theory. The K0-5 approximation almost exactly describes the internal SSS at smooth loads. The Reissner variational principle yields comparable results with Hu-Vashitsu's variational principle. The Lagrangian variational principle produces less accurate results.

Our research points to the need to use a variant of MT to determine the SSS of thick plates, plates of medium thickness, as well as thin plates at loads, which lead to a large gradient of change of SSS.

Table 2

Components of the SSS of a square transropic plate

($h/a = 0,2;\ G'/G = 0,1;\ E'/E = 1;\ \nu' = \nu = 0,3;\ m = n = 1$ **)**

$\frac{z}{h}$	**К1 К01**	**К13 К0-3**	**К135 К0-5**	**ES**	Δ_{t5} **%**	Δ_{t3} **%**	Δ_{t1} **%**	Δ_{pq} **%**
	$\tilde{\sigma}_x$							
0,5	-5,074 -5,247	-6,466 -6,555	-6,589 -6,680	-6,593 -6,685	0,06 0,07	1,93 1,94	23,4 21,5	1,38 -
0	0 -0,1725	0 -0,0679	0 -0,0681	0 -0,0685	- -	- -	- -	- -
-0,5	5,074 4,902	6,466 6,377	6,589 6,498	6,593 6,502	- 0,06	- 1,92	- 24,6	1,40 -
	$\tilde{W}$							
0,5	-56,75 -56,75	-55,26 -55,49	-55,16 -55,38	-55,15 -55,38	0,02 0,00	0,20 0,20	2,90 2,47	0,42 -
0	-56,75	-55,71	-55,70	-55,70	0,00	0,02	1,89	-
-0,5	-56,75 -56,75	-55,26 -55,04	-55,16 -54,93	-55,15 -54,92	- 0,02	0,22	3,33	0,42 -

Table 3

$\tilde{\sigma}_x$ **stress in square isotropic plate (** $z/h = 0,5;\ m = n$ **)**

m	a/h	$\tilde{\sigma}_x$			
		CT	**К13**	**К135**	**ET**
1	1	-0,1976	-0,4440	-0,4743	-0,4760
	2	-0,7903	-1,013	-1,018	-1,019
	5	-4,940	-5,148	-5,149	-5,152
	10	-19,76	-19,96	-19,96	-20,02
5	2	-0,0316	-0,2134	-0,3501	-0,4002
	5	-0,1976	-0,4440	-0,4743	-0,4760
	10	-0.7903	-1.013	-1.018	-1.019
	20	-3.161	-3.371	-3.373	-3.373
9	2	-0,0098	-0,1041	-0,2267	-0,4000
	5	-0,0610	-0,2890	-0,3864	-0,4035
	10	-0,2439	-0,4872	-0,5108	-0,5119
	20	-0,9757	-1,195	-1,200	-1,200

Table 4

$\tilde{W}$ displacement in square isotropic plate ($z/h = 0,5;\ m = n$)

m	a/h	$\tilde{W}$			
		CT	**K13**	**K135**	**ET**
1	1	-0,0280	-0,2508	-0,2372	-0,2343
	2	-0,4485	-1,014	-0,9946	-0,9947
	5	-17,52	-20,41	-20,39	-20,41
	10	-280,3	-291,5	-291,5	-292,3
5	2	-0,0007	-0,0738	-0,0899	-0,0820
	5	-0,0280	-0,2508	-0,2372	-0,2343
	10	-0.4485	-1.014	-0.9946	-0.9946
	20	-7.176	-9.071	-9.051	-9.054
9	2	-0,0001	-0,0271	-0,0447	-0,0455
	5	-0,0027	-0,1173	-0,1220	-0,1145
	10	-0,0427	-0,2944	-0,2791	-0,2769
	20	-0,6836	-1,353	-1,334	-1,334

4. Transformation of the system of differential equations of equilibrium. Definitive system of differential equations of boundary value problems

In paragraphs 4.1 and 4.2 below, we will outline some equations and relations [53, p. 84; 56, p.127] that we will need later in order to analytically integrate the system of DE of equilibrium for skew-symmetric deformation.

4.1. The transformation of the system of differential equations of equilibrium in the approximation of K13… N $(N \geq 3)$. The boundary conditions on the upper and lower boundary planes of the plate have the form (2.2). The system DE (2.8) in the approximation K13… N (N is an odd positive integer $(N \geq 3)$) has order $3(N+1)$.

After algebraic and differential transformations of system (2.8), two DE systems are obtained. One system (homogeneous system of order $(N+1)$)) describes the boundary effect of the vortex:

$$\sum_{j=1,3}^{N} H_{ij}\,\psi_j = 0;\ (i = 1,\ 3, ..., N), \tag{4.1a}$$

where

$$\psi_j(x, y) = \partial u_j / \partial y - \partial v_j / \partial x\,. \tag{4.1b}$$

In (4.1a) H_{ij} $(j = 1, 3, ..., N)$, $(i = j)$ are second-order differential operators; H_{ij} $(j = 1, 3, ..., N)$, $(i \neq j)$ are zero-order differential operators; $\psi_j(x, y)$ – vortex functions, the operators depend on the MGP.

Another DE system (inhomogeneous $2(N+1)$ order system) determines the internal SSS and the potential boundary effect with respect to $w_i(x, y)$, $(i = 1, 3, ..., N)$ functions:

$$\sum_{i=1,3}^{N} \Pi_{ji} w_i(x, y) = \Pi_{jq} q(x, y) \quad (j = 1, 3, ..., N), \tag{4.2}$$

where Π_{ji} are fourth-order differential operators. Π_{jq} –second-order differential operators. In the approximation K13 (N = 3), the differential operators of equations (4.2) Π_{ji} $(i, j = 1, 3)$ and Π_{jq} have the following form:

$$\Pi_{11} = \mu_{114}\nabla^4 + \mu_{112}\nabla^2; \; \Pi_{13} = \mu_{134}\nabla^4 + \mu_{32}\nabla^2 + \mu_{130}; \tag{4.3}$$

$$\Pi_{31} = \mu_{314}\nabla^4 + \mu_{312}\nabla^2; \; \Pi_{33} = \mu_{334}\nabla^4 + \mu_{332}\nabla^2 + \mu_{330};$$

$$\Pi_{1q} = \mu_{12}\nabla^2 - \mu_{10}; \; \Pi_{3q} = \mu_{32}\nabla^2 - \mu_{30}; \; \nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2,$$

where $\propto_{114}, \propto_{112}, ..., \propto_{30}$ –MGP, ∇^2 –the Laplace operator.

4.2. The determining system of DE in the approximation K13…N. Forms of general solutions.

4.2.1. Forms of general solutions for the system (4.1a). General solutions of the system of DE of vortex boundary effect (4.1a) are given in the form:

$$\psi_i(x, y) = H_{1i}^0 \, \psi(x, y), \; (i = 1, 3, ..., N; \; N \geq 3), \tag{4.4}$$

where H_{1i}^0 is the adjuncts of the determinant of H_0 of the system (4.1a).

The determining equation (4.4) for finding the function $\psi(x, y)$ is a homogeneous DE of order $(N+1)$:

$$H_0 \, \psi(x, y) = 0, \tag{4.5}$$

where

$$H_0 = (\nabla^2 - k_1)(\nabla^2 - k_2) \cdot \ldots \cdot (\nabla^2 - k_{(N+1)/2}),$$

k with indexes–MGP.

The general solution of differential equation (4.5) is represented by:

$$\psi(x, y) = \sum_{i=1}^{(N+1)/2} \psi^{(i)}(x, y), \tag{4.6}$$

where $\psi^{(i)}(x, y)$ are the general solutions of the Helmholtz differential equations:

$$(\nabla^2 - k_i)\ \psi^{(i)}(x, y) = 0,\ \ (i = 1, 2, ..., (N+1)/2;\ N \geq 3)\ . \qquad (4.7)$$

These solutions are expressed through Bessel functions, which depends on the constants of k_i . Thus, the vortex functions of $\psi_i(x, y)$ are determined from the dependences (4.4) taking into account (4.6) and the general solutions of equations (4.7).

4.2.2. Forms of general solutions for the system (4.2). Forms of general solutions of system (4.2) are obtained by the operator method and have the form:

$$w_i(x, y) = \sum_{k=1,3}^{N} \varPi_{k\,i}^{0} \varPhi_k(x, y), (\,i = 1,\ 3, ..., N;\ N \geq 3), \qquad (4.8)$$

where $\varPi_{k\,i}^{0}$ is the adjuncts of the determinant of the $\varPi_0$ system (4.2), $\varPhi_k(x, y)$ is the new sought-after functions.

Based on (4.2), (4.8), we obtain a defining system of DE for functions $\varPhi_k(x, y)$, which describes the internal SSS and the potential boundary effect:

$$D_0\ D_0\ D_1\ D_2 \cdot ... \cdot D_{(N-1)} \varPhi_k\ (x, y) = a_{k\,0}\ D_{k\ 0}\ q(x, y); \qquad (4.9)$$

$$k = 1, 3, ..., N;\ N \geq 3,$$

where

$$D_0 = \nabla^2;\ D_i = \nabla^2 - s_i; D_{k\,0} = \nabla^2 - s_{k\,0};\ \ i = 1, 2, ..., N-1;$$

$s_i, s_{k\,0}, a_{k\,0}$ – MGP; s_i – the roots of the corresponding characteristic equation (for transversely isotropic plates with small shear stiffness $s_i \rangle\ 0$).

DE (4.9) are more convenient than (4.2), since the left-hand sides of the equations are the same.

The forms of general solutions of system (4.9) are as follows:

$$\varPhi_1(x, y) = \varPhi_{1B}(x, y) + \varPhi_{1\varPi}(x, y) + \varPhi_{1r}(x, y); \qquad (4.10a)$$

$$\varPhi_k(x, y) = \varPhi_{kr}(x, y),\ \ (k = 3, 5, ..., N)\ . \qquad (4.10b)$$

In formulas (4.10a), (4.10b):

$\varPhi_{1B}$ is the general solution of a harmonic equation

$$\nabla^4 \varPhi_1 = 0\ ; \qquad (4.11)$$

$\varPhi_{1\varPi}$ is the general solution of a homogeneous DE of order $2(N-1)$

$$D_1\ D_2 \cdot ... \cdot D_{(N-1)} \varPhi_k\ (x, y) = 0;\ k = 1, 3, ..., N;\ N \geq 3; \qquad (4.12)$$

$\varPhi_{k\,r}\ (x, y)$ – partial solutions of inhomogeneous DE (4.9).

The potential boundary effect is described by the general solution $\varPhi_{1\varPi}$ of homogeneous equation (4.12). The general solution of $\varPhi_{1B}$ of equation (4.11) together with the partial solutions of Φ_{kr} $(k = 1, 3, ..., N)$ of inhomogeneous DE (4.9) of order $3(N+1)$ describe the internal SSS of the plates. Thus, the equations of the internal SSS and the potential boundary effect are separated.

Forms of general solutions for $w_k(x, y)$ based on (4.8), (4.10a), (4.10b) have the form:

$$w_i(x, y) = \Pi^0_{1i}(\Phi_{1B}(x, y) + \Phi_{1\Pi}(x, y)) + \sum_{k=1,3}^{N} \Pi^0_{k\,i}\Phi_{k\,r}(x, y), \qquad (4.13)$$

$$(i = 1,\ 3, ..., N).$$

The components u_k, v_k $(k = 1, 3, ..., N)$ are determined from the transformed equations of equilibrium (2.8):

$$u_k(x, y) = \sum_{i=1,3}^{N} (\lambda_{k\phi\,i}\phi_{i,x} + \lambda_{k\psi\,i}\psi_{i,y} + \lambda_{k\,w\,i}w_{i,x}) + \lambda_{kq}q_{,x}, \qquad (4.14)$$

$$(u_k, \phi_{,x}, \psi_{,y}, w_{,x}, q_{,x} \to v_k, \phi_{,y}, -\psi_{,x}, w_{,y}, q_{,y}),$$

where

$$\phi_i(x, y) = \lambda_{i1}\nabla^2 w_1 + \sum_{k=3,5}^{N} (\lambda_{ik}\nabla^2 + \lambda'_{ik})w_k + \lambda_{i\,q}q; \qquad (4.15)$$

$$\phi_i(x, y) = \partial u_i \,/\, \partial x + \partial v_i \,/\, \partial y,$$

λ with indexes – MGP.

5. Method of reducing inhomogeneous high-order differential equations to low-order inhomogeneous equations

Consider the system of inhomogeneous equations (4.9), which has a high order $2(N+1)$:

$$D_0\, D_0\, D_1\, D_2 \cdot ... \cdot D_{(N-1)}\Phi_k\,(x, y) = a_{k\,0}\, D_{k\,0}\, q(x, y); \qquad (4.9)$$

$$k = 1, 3, ..., N;\ N \geq 3.$$

In (4.9) s_i $(i = 1, 2, ..., N-1)$ – the roots of the corresponding characteristic equation.

The difficulty of finding partial solutions to equations (4.9) depends on the complexity of the right-hand sides and the order of the equations. Special complications arise if the right-hand sides of the equations have discontinuous functions that correspond to local and concentrated load. Then analytical finding of partial solutions using integral transformations is either very complicated or impossible. In such cases, you can use infinite series, numerical methods, or combined numerical-analytic methods. Therefore, the goal is to simplify the definition of partial and, consequently, general solutions of differential equations (4.9).

The simplification of the search for partial solutions can be achieved by applying the method of splitting an inhomogeneous high-order equation into several inhomogeneous low-order equations [58, p. 154; 59, p. 60]. It is shown that the classical operator method for integrating a non-homogeneous DE follows from the aforementioned splitting method.

In the publication [29, p. 125] used the classical operator method of integrating non-homogeneous fourth-order differential equations of the $(\nabla^2 - s_1)(\nabla^2 - s_2)\Phi(x, y) = (\nabla^2 - s_0) f(x, y)$ type, which led to two inhomogeneous Helmholtz equations. In the future, integral transformations to the obtained equations were not applied. In the article [45, p. 382] a partial solution of the basic equation $\nabla^2\nabla^2\sigma(r) + s_1\nabla^2\sigma(r) = a_0\, q(r)$ of the theory of thin isotropic spherical shells of small curvature was found by direct application of the integral Hankel transform [49], which has already led to some complications. The analysis shows that the use of the methods of integral transformations directly in higher-order equations leads to considerable mathematical difficulties, since in the reverse transformation it is ne-cessary to find cumbersome integrals with parameters that are not given in the well-known literature [12].

5.1. The idea of the method of finding partial solutions

The method is based on the complex use of the operator method and the method of integral transformations. It consists of the following:

1) a partial solution of a non-homogeneous DE of high order using the operator integration method is represented as a differential operator from a linear combination of partial solutions of inhomogeneous DE of a low order (second and fourth);

2) the partial solution of each inhomogeneous equation of low order, indicated in 1, is determined by the methods of integral transforma-tions;

3) with allowance for 1 and 2, a partial solution of a non-homogeneous equation of a high order is determined.

5.2. Representation of a partial solution of an inhomogeneous high-order equation through partial solutions of inhomogeneous equations of a small order

We use the operator method to integrate higher order differential equations, which are decisive in the considered variant of the mathematical theory of plates. Integrating (4.9), after some transformations, we obtain formulas for partial solutions of the considered class of inhomogeneous equations, which can be expressed in terms of partial solutions of inhomogeneous second and fourth order equations. We give these formulas.

5.2.1. Differential equations of the eighth order.

$$D_0\, D_0\, D_1\, D_2\, \varPhi_k\,(x,y) = a_{k\,0}\, D_{k\,0}\, q(x,y),\ (k=1,3)\,. \tag{5.1}$$

The partial solutions of the $\varPhi_{k\,r}(x,y)$ equation (6) have the form:

$$\varPhi_{k\,r}(x,y) = \frac{a_{k\,0}D_{k\,0}}{s_1 s_2}\left(\frac{s_2}{s_1 s_{12}}(f_{1r} - f_{0r}) + \frac{s_1}{s_2 s_{21}}(f_{2r} - f_{0r}) + f_{00\,r}\right), \tag{5.2}$$

where here and below $s_{i\,j} = s_i - s_j$;

$f_{0r}(x,y)$ – the partial solution of the differential Poisson's equation

$$D_0\ f_0(x,y) \equiv \nabla^2 f_0(x,y) = q(x,y)\,; \tag{5.3}$$

$f_{00r}(x,y)$ – the partial solution of a inhomogeneous equation of the fourth order

$$D_0 D_0 f_{00}(x,y) \equiv \nabla^4 f_{00}(x,y) = q(x,y)\,; \tag{5.4}$$

$f_{i\,r}(x,y)$ – the partial solutions of the inhomogeneous Helmholtz differential equations

$$D_i\ f_i(x,y) \equiv (\nabla^2 - s_i) f_i(x,y) = q(x,y),\ \ (i=1,2)\,. \tag{5.5}$$

5.2.2. Differential equations of the twelfth order.

$$D_0\, D_0\, D_1\, D_2 D_3\, D_4\, .\varPhi_k\,(x,y) = a_{k\,0}\, D_{k\,0}\, q(x,y),\ (k=1,3,5). \tag{5.6}$$

Partial solutions of the $\varPhi_{k\,r}(x,y)$ equation (5.6) are presented thus:

$$\begin{aligned} &\varPhi_{k\,r}(x,y) = \\ &= \frac{a_{k\,0}D_{k\,0}}{s_1 s_2 s_3 s_4}\Big(\frac{s_2 s_3 s_4}{s_1 s_{12} s_{13} s_{14}}(f_{1r} - f_{0r}) + \frac{s_1 s_3 s_4}{s_2 s_{21} s_{23} s_{24}}(f_{2r} - f_{0r}) + \\ &+ \frac{s_1 s_2 s_4}{s_3 s_{31} s_{32} s_{34}}(f_{3r} - f_{0r}) + \frac{s_1 s_2 s_3}{s_4 s_{41} s_{42} s_{43}}(f_{4r} - f_{0r}) + f_{00r}\Big), \end{aligned} \tag{5.7}$$

where $f_{0\,r}(x,y), f_{00\,r}(x,y), f_{i\,r}(x,y)\ (i=1,2,3,4)$ – are partial solutions of the corresponding differential equations (5.3)–(5.5).

5.2.3. Differential equations (4.9) of the order of $2(N+1)$. Analyzing the structure of the specific solutions obtained, depending on the order of the equations, we obtain partial solutions of the DE (4.9) of the order of $2(N+1)$ in the form:

$$\Phi_{kr}(x,y)=\frac{a_{k0}D_{k0}}{s_1s_2\ldots s_{2n-2}}\left(\frac{s_2s_3\ldots s_{2n-2}}{s_1s_{1,2}s_{1,3}\ldots s_{1,2n-2}}(f_{1,r}-f_{0r})+\right.$$

$$+\frac{s_1s_3s_4\ldots s_{2n-2}}{s_2s_{2,1}s_{2,3}\ldots s_{2,2n-2}}(f_{2,r}-f_{0r})+\ldots+ \qquad (5.8)$$

$$\left.+\frac{s_1s_2s_3\ldots s_{2n-3}}{s_{2n-2}s_{2n-2,1}s_{2n-2,2}\ldots s_{2n-2,2n-3}}(f_{2n-2,r}-f_{0r})+f_{00r}\right),$$

where $s_{m,l}=s_m-s_l$; $f_{i,r}(x,y)$ $(i=1,2,...,N-1)$ – are partial solutions of DE (5.5).

Putting in (5.8) $n=2$ and $n=3$, we get the formulas (5.2) and (5.7). We will present the use of the proposed method for finding partial solutions when integrating the system of differential bending equations of the plate (5.1) in the approximation of K13. Consider the action of axisymmetric intermittent loads.

System (5.1) consists of two inhomogeneous DE of the eighth order.

Consider the axisymmetric problem of plate bending in the K13 approximation. Vortex edge effect will be absent. We apply the Hankel integral transformation. We also find the general solutions of the system of equations (5.1). This will allow us to find general solutions for all components of the stress-strain state of the plate. Analytical solutions for different classes of boundary-value problems will be found taking into account boundary conditions on the side surface.

6. Transverse loading of the plate around the circumference

In this case, the load $q(r)$ is represented as follows:

$$q(r)=q_0\,\delta(r-r_0)\,, \qquad (6.1)$$

where $q_0=const$, $\delta(r-r_0)$ is the Dirac delta function:

$$\delta(r-r_0)=\begin{cases}0, & (r\neq r_o);\\ \infty, & (r=r_o).\end{cases}$$

Then the differential equations (5.1), (5.3)–(5.5) take the form:

$$D_0\,D_0\,D_1\,D_2\,\Phi_k\,(r)=a_{k0}\,D_{k\,0}\,q(r),\ (k=1,3)\,; \qquad (6.2)$$

$$D_0\,f_0(r)\equiv\nabla^2f_0(r)=q(r)\,; \qquad (6.3)$$

$$D_0D_0f_{00}(r)\equiv\nabla^4f_{00}(r)=q(r)\,; \qquad (6.4)$$

$$D_i\,f_i(r)\equiv(\nabla^2-s_i)f_i(r)=q(r),\ (i=1,2)\,, \qquad (6.5)$$

where $q(r)$ satisfies the dependence (6.1), and the Laplace operator has the form: $\nabla^2 = d^2 / d\,r^2 + (1/r)\,d / dr$.

Partial solutions of equation (6.2) are found in accordance with (5.2), and the partial solutions of equations (6.3)–(6.5) are sought by the Hankel integral transformation method [49, p. 67].

We give the main formulas in the method of the integral Hankel transform.

The image of the $f(r)$ function in the Hankel transform is the $F(p)$ function:

$$F(p) = \int_0^\infty rJ_0(pr)f(r)dr\ , \tag{6.6}$$

where $J_0(pr)$ is the Bessel function of the first kind of zero order [22].

Other formulas:

$$\int_0^\infty rJ_0(pr)\,\nabla^2 f(r)dr = -p^2 F(p)\ ;$$

$$\int_0^\infty rJ_0(pr)\,\nabla^{2m} f(r)dr = (-1)^m p^{2m} F(p),\ \ m \in N\ ; \tag{6.7}$$

$$\int_0^\infty rJ_0(pr)\,\delta(r - r_0)dr = r_0 J_0(pr_0)\ . \tag{6.8}$$

The original function $f(r)$ is determined from its image (by function $F(p)$) as follows:

$$f(r) = \int_0^\infty p\,J_0(p\,r)F(p)dp\ . \tag{6.9}$$

We assume that the functions on the left-hand sides of equations (6.3)-(6.5) satisfy the conditions of the Hankel integral transformation, namely:

$$r\,F(r) \to 0,\ r\,F'(r) \to 0\ , \ldots,\text{if } r \to 0,\ r \to \infty\ ,$$

where $F(r)$ is any of the functions of the left side of equations (6.3)–(6.5); the function $q_0\,\delta(r - r_0)$ on the right-hand side of (6.3)–(6.5) satisfies these conditions.

6.1. Partial solutions of equations (6.3), (6.4) under load (6.1)

Without dwelling on the solution of the inhomogeneous differential equations (6.3) and (6.4), we give them partial solutions.

Partial solutions of equation (6.3):

$$f_{0r}(r) = q_0 r_0 \ln r_0\ ,\ \ (r \langle r_0\,)\ ; \tag{6.10}$$

$$f_{0r}(r) = q_0 r_0 \ln r\ ,\ \ (r \rangle r_0\,)\ . \tag{6.11}$$

Partial solutions of equation (6.4):

$$f_{00r}(r) = \frac{q_0 r_0}{4}((r^2 + r_0^2)\ln r_0 + r^2), \ \ (r \langle r_0 \ ); \qquad (6.12)$$

$$f_{00r}(r) = \frac{q_0\, r_0}{4}((r^2 + r_0^2)\ln r + r_0^2), \ \ (r \rangle r_0 \ ). \qquad (6.13)$$

By direct verification, one can see that (6.10)–(6.13) are indeed partial solutions of the differential equations (6.3) and (6.4). Here, the dimensions r and r_0 are understood to be dimensionless, which are numerically equal to their dimensional values. Taking into account the boundary conditions, the solution of the boundary value problems will include the relations of dimensional quantities, which will already be dimensionless.

6.2. Partial solutions of differential equations (6.5) under loading (6.1)

Equations (6.5) in the image space have the following form:

$$\int_0^\infty rJ_0(pr)(\nabla^2 - s_i) f_i(r)\, dr = \int_0^\infty rJ_0(pr)\, q_0\, \delta(r - r_0)\, dr, \ \ (i = 1, 2).$$

Therefore, taking (6.6)-(6.8) into account, we find:

$$F_i(p) = \frac{-q_0 r_0}{p^2 + s_i} J_0(p\, r_0).$$

Returning to $f_{ir}(r)$, we get:

$$f_{ir}(r) = \int_0^\infty pJ_0(pr)\, F_i(p) dp = -q_0\, r_0 \int_0^\infty \frac{p\, J_0(pr) J_0(pr_0)}{p^2 + s_i} dp. \qquad (6.14)$$

Taking into account the formulas for the last integral given in the reference book [12, p. 693], we have:

$$\int_0^\infty \frac{p\, J_0(pr) J_0(pr_0)}{p^2 + s_i} dp = I_0(r\sqrt{s_i}) K_0(r_0\sqrt{s_i}), \ \ (r \langle r_0);$$

$$\int_0^\infty \frac{p\, J_0(pr) J_0(pr_0)}{p^2 + s_i} dp = I_0(r_0\sqrt{s_i}) K_0(r\sqrt{s_i}), \ \ (r \rangle r_0),$$

where I_0 is a modified Bessel function of zero order, K_0 is a zero-order Macdonald function.

Consequently, the partial solutions $f_{ir}(r)$ of equations (6.5) taking into account (6.14) and the last equalities take the following form:

$$f_{ir}(r) = -q_0\, r_0\, I_0(r\sqrt{s_i}) K_0(r_0\sqrt{s_i}), \ \ (r \langle r_0); \qquad (6.15)$$

$$f_{ir}(r) = -q_0\, r_0\, I_0(r_0\sqrt{s_i}) K_0(r\sqrt{s_i}), \ \ (r \rangle r_0). \qquad (6.16)$$

6.3. Partial solutions of differential equations (6.2)

We obtain partial solutions of DE (6.2) taking into account (5.2), (6.1), (6.10)–(6.13), (6.15), (6.16):

$$\Phi_{kr}(r) = -\frac{a_{k0}\, q_0\, r_0\, D_{k0}}{s_1\, s_2}\Big(\frac{s_2}{s_1 s_{12}} I_0(r\sqrt{s_1})K_0(r_0\sqrt{s_1}) + \frac{s_1}{s_2 s_{21}} I_0(r\sqrt{s_2})K_0(r_0\sqrt{s_2}) + s_{12}^0 \ln r_0 - \frac{1}{4}((r^2 + r_0^2)\ln r_0 + r^2)\Big), \quad (r \langle r_0,\ \ k = 1,3); \tag{6.17}$$

$$\Phi_{kr}(r) = -\frac{a_{k0}\, q_0\, r_0\, D_{k0}}{s_1\, s_2}\Big(\frac{s_2}{s_1 s_{12}} I_0(r_0\sqrt{s_1})K_0(r\sqrt{s_1}) + \frac{s_1}{s_2 s_{21}} I_0(r_0\sqrt{s_2})K_0(r\sqrt{s_2}) + s_{12}^0 \ln r - \frac{1}{4}((r^2 + r_0^2)\ln r + r_0^2)\Big), \quad (r \rangle r_0,\ \ k = 1,3), \tag{6.18}$$

where $s_{12}^0 = \dfrac{s_2}{s_1 s_{12}} + \dfrac{s_1}{s_2 s_{21}}$.

We take into account the following relations [22]:

$$\nabla^2 K_0(\sqrt{s_i}\, r) = s_i K_0(\sqrt{s_i}\, r),\ \nabla^4 K_0(\sqrt{s_i}\, r) = s_i^2 K_0(\sqrt{s_i}\, r),$$

$$\nabla^2 I_0(\sqrt{s_i}\, r) = s_i I_0(\sqrt{s_i}\, r),\ \nabla^4 I_0(\sqrt{s_i}\, r) = s_i^2 I_0(\sqrt{s_i}\, r),\ (i = 1,\ 2).$$

Then the partial solutions (6.17) and (6.18) of equation (6.2) take the following form:

$$\Phi_{kr}(r) = -\frac{a_{k0}\, q_0\, r_0}{s_1\, s_2}\Big(\frac{s_2}{s_1 s_{12}}(s_1 - s_{k0}) I_0(r\sqrt{s_1})K_0(r_0\sqrt{s_1}) + \frac{s_1}{s_2 s_{21}}(s_2 - s_{k0}) I_0(r\sqrt{s_2})K_0(r_0\sqrt{s_2}) - \ln r_0 - 1 + s_{k0}(-s_{12}^0 \ln r_0 + \frac{1}{4}(r^2 + r_0^2)\ln r_0 + \frac{1}{4}r^2)\Big), \quad (r \langle r_0,\ \ k = 1,3); \tag{6.19}$$

$$\Phi_{kr}(r) = -\frac{a_{k0}\, q_0\, r_0}{s_1\, s_2}\Big(\frac{s_2}{s_1 s_{12}}(s_1 - s_{k0}) I_0(r_0\sqrt{s_1})K_0(r\sqrt{s_1}) + \frac{s_1}{s_2 s_{21}}(s_2 - s_{k0}) I_0(r_0\sqrt{s_2})K_0(r\sqrt{s_2}) - \ln r - 1 + s_{k0}(-s_{12}^0 \ln r + \frac{1}{4}(r^2 + r_0^2)\ln r + \frac{1}{4}r_0^2)\Big), \quad (r \rangle r_0,\ \ k = 1,3). \tag{6.20}$$

Pairing conditions for $\Phi_{kr}(r)$, if $r = r_0$, are met.

7. Transverse loading of the plate along the ring

Consider a plate under the action of a uniformly distributed load q_0 along a circular ring with radii r_1 and r_2 $(r_1 \langle r_2)$:

$$q(r) = \begin{cases} 0, & (r \langle r_1); \\ q_0, & (r_1 \langle r \langle r_2); \\ 0, & (r \rangle r_2). \end{cases} \qquad (7.1)$$

To obtain partial solutions of the differential equations (6.2) with allowance for (7.1), we replace q_0 in (6.19) and (6.20) by $q_0\,dr_0$ and integrate them in the range from r_1 to r_2. We obtain:

$$\begin{aligned} &\Phi_{kr}(r) = \\ &= -\frac{a_{k0}q_0}{s_1\,s_2}\Big(\frac{s_2}{s_1 s_{12}\sqrt{s_1}}(s_1 - s_{k0})I_0(r\sqrt{s_1})(r_1K_1(r_1\sqrt{s_1}) - r_2K_1(r_2\sqrt{s_1})) + \\ &+ \frac{s_1}{s_2 s_{21}\sqrt{s_2}}(s_2 - s_{k0})I_0(r\sqrt{s_2})(r_1K_1(r_1\sqrt{s_2}) - r_2K_1(r_2\sqrt{s_2})) - \frac{1}{4}(r_2^2 - r_1^2) + \\ &+ \frac{1}{2}\Big(\frac{s_{k0}r^2}{4} - 1 - s_{k0}s_{12}^0\Big)(r_2^2\ln r_2 - r_1^2\ln r_1) + \frac{s_{k0}}{16}(r_2^4\ln r_2 - r_1^4\ln r_1 + \\ &+ 4s_{12}^0(r_2^2 - r_1^2) - \frac{1}{4}(r_2^4 - r_1^4) + r^2(r_2^2 - r_1^2))\Big), \quad (r \langle r_1, \ k = 1,3); \end{aligned} \qquad (7.2)$$

$$\begin{aligned} &\Phi_{kr}(r) = \\ &= -\frac{a_{k0}q_0}{s_1\,s_2}\Big(\frac{s_2}{s_1 s_{12}\sqrt{s_1}}(s_1 - s_{k0})K_0(r\sqrt{s_1})(r_2I_1(r_2\sqrt{s_1}) - r_1I_1(r_1\sqrt{s_1})) + \\ &+ \frac{s_1}{s_2 s_{21}\sqrt{s_2}}(s_2 - s_{k0})K_0(r\sqrt{s_2})(r_2I_1(r_2\sqrt{s_2}) - r_1I_1(r_1\sqrt{s_2})) - \\ &- \frac{1}{2}(1 + \ln r + s_{k0}s_{12}^0\ln r)(r_2^2 - r_1^2) + \frac{s_{k0}}{16}(2r^2(r_2^2 - r_1^2)\ln r + (r_2^4 - r_1^4)(1 + \ln r))\Big), \\ &(r \rangle r_2, k = 1,3), \end{aligned} \qquad (7.3)$$

I_1, K_1 is the modified first-order Bessel and Macdonald functions.

To find partial solutions of the differential equations (6.2) for the load (7.1) with $r_1 \langle r \langle r_2$, we need to add the right-hand sides of (7.2) and (7.3). In this case, in (7.2), it is necessary to replace r_1 by r, and in (7.3)– r_2 to r. We get

$$\Phi_{kr}(r) = -\frac{a_{k0}q_0}{s_1 s_2}(A_{kr}(r) + B_{kr}(r))\,; \qquad (7.4)$$

$$\begin{aligned} A_{kr}(r) &= \frac{s_2}{s_1 s_{12}\sqrt{s_1}}(s_1 - s_{k0})I_0(r\sqrt{s_1})(r\,K_1(r\,\sqrt{s_1}) - r_2K_1(r_2\sqrt{s_1})) + \\ &+ \frac{s_1}{s_2 s_{21}\sqrt{s_2}}(s_2 - s_{k0})I_0(r\sqrt{s_2})(r\,K_1(r\,\sqrt{s_2}) - r_2K_1(r_2\sqrt{s_2})) + \\ &+ \frac{1}{4}(r_2^2 - r^2)(s_{k0}s_{12}^0 - 1) + \frac{1}{2}\Big(\frac{s_{k0}r^2}{4} - 1 - s_{k0}s_{12}^0\Big)(r_2^2\ln r_2 - r^2\ln r); \end{aligned}$$

$$B_{kr}(r) = \frac{s_{k0}}{16}(r_2^4 \ln r_2 - r^4 \ln r - \frac{1}{4}(r_2^4 - r^4) + r^2(r_2^2 - r^2)) +$$
$$+ \frac{s_2}{s_1 s_{12}\sqrt{s_1}}(s_1 - s_{k0})K_0(r\sqrt{s_1})(r\, I_1(r\sqrt{s_1}) - r_1 I_1(r_1\sqrt{s_1})) +$$
$$+ \frac{s_1}{s_2 s_{21}\sqrt{s_2}}(s_2 - s_{k0})K_0(r\sqrt{s_2})(r\, I_1(r\sqrt{s_2}) - r_1 I_1(r_1\sqrt{s_2})) -$$
$$-\frac{1}{2}(1 + \ln r + s_{k0}s_{12}^0 \ln r)(r^2 - r_1^2) + \frac{s_{k0}}{16}(2r^2(r^2 - r_1^2)\ln r + (r^4 - r_1^4)(1 + \ln r))) ,$$
$$(r_1 \langle r \langle r_2) .$$

The conjugation conditions for $r = r_1$ and $r = r_2$ are satisfied.

8. Transverse loading of the plate on the area of the circle

Let us now consider a plate of arbitrary constant thickness that undergoes a transverse load q_0 uniformly distributed along a circular pad of radius r_2 :

$$q(r) = \begin{cases} q_0, & (r \langle r_2); \\ 0, & (r \rangle r_2). \end{cases} \tag{8.1}$$

Partial solutions of the differential equations (6.2) with allowance for (8.1) are obtained if we put $r_1 = 0$ in expressions (7.3) and (7.4). Then from (7.4) we find $\Phi_{kr}(r)$ for $r \langle r_2$:

$$\Phi_{kr}(r) = -\frac{a_{k0}q_0}{s_1 s_2}(C_{kr}(r) + D_{kr}(r)); \tag{8.2}$$

$$C_{kr}(r) = \frac{s_2}{s_1 s_{12}\sqrt{s_1}}(s_1 - s_{k0})I_0(r\sqrt{s_1})(r\, K_1(r\sqrt{s_1}) - r_2 K_1(r_2\sqrt{s_1})) +$$
$$+ \frac{s_1}{s_2 s_{21}\sqrt{s_2}}(s_2 - s_{k0})I_0(r\sqrt{s_2})(r\, K_1(r\sqrt{s_2}) - r_2 K_1(r_2\sqrt{s_2})) +$$
$$+\frac{1}{4}(r_2^2 - r^2)(s_{k0}s_{12}^0 - 1) + \frac{1}{2}(\frac{s_{k0}r^2}{4} - 1 - s_{k0}s_{12}^0)(r_2^2 \ln r_2 - r^2 \ln r\,);$$

$$D_{kr}(r) = \frac{s_{k0}}{16}(r_2^4 \ln r_2 - r^4 \ln r - \frac{1}{4}(r_2^4 - r^4) + r^2(r_2^2 - r^2)) +$$
$$+ \frac{s_2}{s_1 s_{12}\sqrt{s_1}}(s_1 - s_{k0})K_0(r\sqrt{s_1})\, r\, I_1(r\sqrt{s_1}) +$$
$$+ \frac{s_1}{s_2 s_{21}\sqrt{s_2}}(s_2 - s_{k0})K_0(r\sqrt{s_2})r\, I_1(r\sqrt{s_2}) -$$
$$-\frac{1}{2}(1 + \ln r + s_{k0}s_{12}^0 \ln r)r^2 + \frac{s_{k0}r^4}{16}(3\ln r + 1)) .$$

From (7.3) we obtain $\Phi_{kr}(r)$ for $r \rangle r_2$:

$$\Phi_{kr}(r) = -\frac{a_{k0}q_0}{s_1\, s_2}\Big(\frac{s_2}{s_1 s_{12}\sqrt{s_1}}(s_1 - s_{k0})K_0(r\sqrt{s_1})r_2 I_1(r_2\sqrt{s_1}) +$$

$$+\frac{s_1}{s_2 s_{21}\sqrt{s_2}}(s_2 - s_{k0})K_0(r\sqrt{s_2})r_2 I_1(r_2\sqrt{s_2}) - \qquad (8.3)$$

$$-\frac{1}{2}(1 + \ln r + s_{k0}s_{12}^0 \ln r)r_2^2 + \frac{s_{k0}}{16}(2r^2 r_2^2 \ln r + r_2^4(1 + \ln r))\Big).$$

The conjugation conditions for solutions $\Phi_{kr}(r)$ for $r = r_2$ according to (8.2) and (8.3) are fulfilled.

9. The plate is loaded with a local load applied to the center

Consider the action of a concentrated force F, applied to the center. Partial solutions of the corresponding differential equations (6.2) in the polar coordinate system will be obtained from (6.20) if r_0 tends to zero. For this, we take into account that $r_0 \to 0$. Then we get

$$I_0(r_0\sqrt{s_i}) \to 1;\;\; 2\pi\, r_0\, q_0 \to F\,;\;\; q_0 \to \frac{F}{2\pi\, r_0}\,;$$

$$\Phi_{kr}(r) =$$

$$= -\frac{a_{k0}\, F}{2\pi\, s_1\, s_2}\Big(\frac{s_2}{s_1 s_{12}}(s_1 - s_{k0})K_0(r\sqrt{s_1}) + \frac{s_1}{s_2 s_{21}}(s_2 - s_{k0})K_0(r\sqrt{s_2}) - \qquad (9.1)$$

$$-(1 + \ln r + s_{k0}(s_{12}^0 - \frac{1}{4}r^2)\ln r)\Big),\;\; (k = 1, 3).$$

Partial solutions (6.2) of the action of a concentrated force in the center can also be found from (8.3) if r_2 is tending to zero. It should be noted that with $r_2 \to 0$:

$$I_1(r_2\sqrt{s_i}) \to \frac{r_2\sqrt{s_i}}{2},\;\; \pi\, r_2^2\, q_0 \to F\,,\;\; q_0 \to \frac{F}{\pi\, r_2^2}\,.$$

It is easy to check that we get the same solutions as (9.1).

Knowing the partial solutions of differential equations (6.2) at constant load, it is possible to obtain partial solutions of the action on the transverse load plate, which varies with r.

10. General solutions of equations (6.2)

General solutions of the $\Phi_{k0}(r)$ of equations of the eighth order (6.2) are represented as:

$$\Phi_{k0}(r) = \Phi_0(r) + \Phi_{kr}(r), \ (k = 1,3) \tag{10.1}$$

where $\Phi_0(r)$ is the general solution of the corresponding homogeneous equation (6.2), and $\Phi_{kr}(r)$ are partial solutions of equations (6.2).

Partial solutions are determined by formulas (6.19) and (6.20) (for a load along the circumference); (7.2), (7.3) and (7.4) (under the ring load); (8.2) and (8.3) (with the load in a circle); by formula (9.1) (under the action of concentrated force in the center).

The general solution $\Phi_0(r)$ of the homogeneous equation (6.2):

$$\Phi_0(r) = f_{00}(r) + f_{10}(r) + f_{20}(r) \,. \tag{10.2}$$

Here

$f_{00}(r)$ is the general solution of the biharmonic equation $\nabla^4 f(r) = 0$:

$$f_{00}(r) = A_0 + B_0 r^2 + C_0 \ln r + D_0 r^2 \ln r \,; \tag{10.3}$$

$f_{i0}(r) \ (i = 1,2)$ – general solutions of the Helmholtz equations $(\nabla^2 - s_i) f_i(r) = 0$:

$$f_{i0}(r) = A_i I_0(r\sqrt{s_i}) + B_i K_0(r\sqrt{s_i}), \quad (i = 1,2). \tag{10.4}$$

In the above formulas $A_0, B_0, ..., B_2$ are constants of integration.

Consequently, the general solutions of equations (6.2) with allowance for (10.1)–(10.4) are defined thus:

$$\begin{gathered}\Phi_{k0}(r) = (A_0 + B_0 r^2 + C_0 \ln r + D_0 r^2 \ln r) + \\ + A_1 I_0(r\sqrt{s_1}) + B_1 K_0(r\sqrt{s_1}) + A_2 I_0(r\sqrt{s_2}) + B_2 K_0(r\sqrt{s_2}) + \Phi_{kr}(r).\end{gathered} \tag{10.5}$$

Based on the general solutions (10.5) of the system of governing equations (6.2), one can find general solutions for all components of displacements and stresses, taking into account the dependences (2.6), (4.3), (4.8), (4.14), (4.15). We must also take into account that in the polar coordinate system for the axisymmetric problem the vortex functions (4.1b) $\psi_i(r) \equiv 0$.

Similarly to the above method, one can find general solutions of the defining system of differential equations (4.9) in higher approximations using (5.7), (5.8), (10.1)–(10.5).

It should be noted that in [46] analytical solutions of axisymmetric problems were found in mathematical series for thick circular and annular plates and short cylinders using three-dimensional equations of elasticity theory. The boundary conditions on the flat surfaces of the slab were met exactly. On a cylindrical surface, static conditions were fulfilled in the integral sense, and ki-nematic conditions were satisfied only on some circumferences of the lateral surface. In our version of the mathematical theory [8, 9, 12, 13, 13d, 48], the boundary conditions on the flat faces of the plate (2.1a), (2.1b) are satisfied exactly, and on the side surface the boundary conditions (2.9)–(2.12) performed exactly across the entire thickness of the plate in every approximation. Therefore, with an increase in the order of approximation, the analytical solution of a variant of the MT will be more accurate than in [46].

The second part of the article presents general solutions of all components of displacements and stresses of the used mathematical theory. The first, second, and third boundary problems are considered for transverse isotropic annular and circular plates of arbitrary thickness for various discontinuous loads. The conditions from which the constants of integration are found are determined. Analytical solutions of various classes of boundary value problems, the nature of their changes in the tangential and transverse directions are investigated.

11. Conclusions

A new method is proposed for integrating inhomogeneous systems of high-order DE of equilibrium, which is based on the following:

1) systems of high-order equilibrium equations are reduced, using algebraic and differential transformations, to convenient defining systems of inhomogeneous differential equations of a high order with respect to new unknown functions;

2) inhomogeneous differential equations of the defining systems of equations are reduced by a method of decreasing the order to inhomogeneous differential equations of the second and fourth orders, the solution of which under intermittent and local loads can be found, in particular, by integral transformation methods;

3) the partial solution of the inhomogeneous equation of the defining high-order system is presented as the differential operator of a linear combination of the partial solutions of the second and fourth order inhomogeneous equations; the general solution of a homogeneous equation of high order is presented as the sum of the general solutions of homogeneous equations of the second and fourth orders;

4) the general solution of a inhomogeneous high-order DE is represented through the general and partiials solutions of the second and fourth order inhomogeneous equations;

5) general solutions of the original systems of DE of high order equilibrium are found by inverse transformations through general and partial solutions of inhomogeneous equations of the second and fourth orders.

The forms of general solutions of inhomogeneous systems of high-order DE of equilibrium in displacements are obtained.

Using the proposed method, general solutions of an inhomogeneous system of eighth-order governing equations of axisymmetric problems of bending transversely isotropic circular and annular plates of arbitrary thickness under intermittent and local loads, which are expressed through the Dirac delta function, are obtained.

Application of the method is presented using the Hankel integral transform. General solutions of the defining systems of inhomogeneous DE are obtained for plates that are under the action of transverse loads acting around a circle, over the area of the ring and over the area of the circle. The action of a concentrated force applied in the center is considered.

Proposed Method:

1) significantly simplifies the finding of particular and general solutions of inhomogeneous systems of high-order differential equations, especially in cases where the right-hand sides of the equations are discontinuous or local functions that correspond to discontinuous or local loads;

2) it is generalized without fundamental mathematical difficulties for solving inhomogeneous systems of differential equations of high order that arise when solving boundary value problems for plates of arbitrary thickness in higher approximations K0-3, K135, K0-5 and others;

3) it can be used to solve boundary problems for plates on a Winkler elastic base, the governing equations of which are of the form:

$$D_1\, D_2 \dots D_n\, \Phi_k\,(x,y) = a_{k\,0}\; D_{k\,0}\; q(x,y)\,,$$

only in this case the general and particular solutions of these inhomogeneous equations are expressed through the general and particular solutions of the inhomogeneous Helmholtz equations;

4) it can also be used in solving inhomogeneous differential equations of class $D_1\, D_2 \dots D_n\, \Phi(x,y) = D\, f(x,y)$ with an arbitrary differential operator D and with differential operators D_i of the form: $D_i = \nabla^2 + b\,\dfrac{\partial}{\partial x} + c\,\dfrac{\partial}{\partial y} + d_i\,,\ \ (i = 1, 2, \dots, n)\,.$

The developed technique for solving boundary-value problems can also be applied to plate theories that use physical and geometric hypotheses.

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