CHAPTER
INNOVATION DYNAMICS OF ECOSYSTEMS AND MATHEMATICAL MODELING OF INNOVATION DIFFUSION

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Summary

An analytical review of the role of the state in moving the economy to an innovative development path as a condition for implementing the concept of sustainable development was conducted. In this regard, in order to select the optimal strategy for innovative development and to predict possible problems that may arise during the implementation of this strategy, it is necessary to construct a mathematical model of the process of innovation diffusion. The paper examines the features of constructing mathematical models that are used to describe economic dynamics, and compares their mathematical apparatus and the scope of application of these models. The most complex, both for construction and for interpretation, is a mathematical model that takes into account the nonlinearity of economic development and the resulting synergistic effects. Using the example of a mathematical model of nonlinear dynamics proposed by the authors, the possibilities of using mathematical methods to study the diffusion of innovations and forecast the development of this process over time are demonstrated. With the help of this model, an analysis of the sustainability of innovation processes was carried out, critical parameters were determined, upon reaching which qualitative changes in the phase trajectories of the movement of the economic system are possible with its transition to a catastrophic mode, namely, the emergence of the Andronov-

Hopf bifurcation and the birth of a limit cycle. An economic interpretation of the obtained results is given.

Introduction

At the current stage of post-industrial economic development, which is classified as the knowledge economy, or knowledge-based economy, it is knowledge, information and innovation that are becoming key factors of production. In turn, human capital is the factor that ensures the possibility of implementing innovative processes, from the development of innovative ideas to their technological implementation, introduction into the production process and the creation of high added value. It is innovation that ensures the transition of the economy to an intensive path of development, replacing a simple increase in resources with a qualitative improvement in production processes and technologies. This creates conditions for the implementation of the concept of sustainable development, according to which the needs of the present generation are fully met, but without compromising the needs of future generations.

The most important priority area for the development of the economy of any state, including the national economy of Ukraine, is the innovative transformation of its production enterprises and the development of human potential with its subsequent transformation into human capital. Accordingly, to ensure the effective implementation of the innovative development strategy, a synthesis of economic and social evolution is necessary, which is accompanied by a synergistic effect. Human capital acts in a company both as a generator of innovations and as a factor ensuring their implementation. The post-industrial economy's focus on innovative development creates favorable conditions for the development of new ideas, opens up new opportunities for technological breakthroughs, and facilitates the implementation of not only economic but also other types of innovations (including political ones), which improve human well-being.

An innovative development path ensures not only stable growth and increased competitiveness of the national economy as a whole and of each individual enterprise. It is a factor that contributes to a significant improvement in the quality of life of the population. Thus, the introduction of new technologies in production not only increases labor productivity, but also creates more comfortable and safe working conditions, and thanks to the use of innovative technologies in everyday life, in education, in medicine, new opportunities for self-realization are opened up for each person.

It should be emphasized that for Ukraine, the search for an effective model of economic development as the implementation of the concept of a knowledge economy is based on the implementation of reforms not only in all sectors of the economy, but such reforms also affect academic science and the educational sphere. In a post-industrial society, education and science represent a strategic resource for the socio-economic, cultural and spiritual development of society as a whole and the creation of conditions for the self-realization of each individual. In this regard, the implementation of such resources ensures an improvement in the well-being of people, contributes to the protection of national interests, strengthening international authority and the formation of a positive image of our state.

Currently, the lack of public financial resources, the depreciation of fixed assets and infrastructure, the alienation and destruction of 20% of the territory as a result of military actions in the east of Ukraine create enormous difficulties in the path of innovative development of the national economy. However, without innovative development, economic recovery in our country is virtually impossible. Therefore, it is necessary to conduct a theoretical analysis of the potential problems that may and will arise during the implementation of innovations. Such analysis is possible thanks to the construction of mathematical models of economic dynamics that describe various development scenarios taking into account the interaction of external factors.

1. The role of the state in the transition to an innovative development

As global experience shows, in modern conditions, sustainable economic growth is impossible without increasing the competitiveness of both individual enterprises and industries, as well as the national economy as a whole. This is due to the transition of post-industrial society to a new stage of development, which is defined as the knowledge economy, or knowledge-based economy, when the production of goods and services is based on the use of high-tech technologies. The concept of knowledge economy was proposed by Fritz Machlup and was developed in the works of Peter Drucker [1; 2 et al.]. Currently, it is generally accepted. In the Age of Discontinuity: Guidelines to Our Changing Society, as Drucker titled his book, he argued that in modern society, capital, natural resources, or even labor are no longer the primary economic resource. The role of such a resource is now assigned to knowledge in the format of human potential with its subsequent transformation into human capital. The company's human capital as its intangible asset, which includes education, training, intelligence and loyalty, is one of the sources of innovative ideas and ensures their implementation [3]. In this regard, in an innovative society, human capital plays the role of a strategic factor ensuring the growth and development of the economic system.

At the current stage of development of the global economic system, the creation of a sustainable infrastructure that promotes comprehensive and sustainable industrialization and innovation is considered one of the tasks of global significance. In September 2015, the United Nations adopted a plan for a shared better future. Among the 17 Sustainable Development Goals [4] formulated within the framework of this plan, Goal number 9 was highlighted. This goal is called "Industry, Innovation and Infrastructure" and aims to build resilient infrastructure, promote inclusive and sustainable industrialization and encourage innovation.

Innovative development is the subject of close monitoring by various global organizations. Thus, the international business school INSEAD, Cornell University, and the World Intellectual Property Organization annually publish an analytical report, The Global Innovation Index, in which all countries of the world are assessed in terms of their development, innovation, and technological sophistication. The Global Innovation Index (GII) is composed of 80 different variables that provide a detailed characterization of the innovative development of countries at different levels of economic development [5]. It reflects the ratio of the costs of implementing innovations (Innovation Input) and the practical effect of implementing innovations (Innovation Output), which allows for an objective assessment of the effectiveness of efforts to develop innovations in a particular country.

In 2013, Ukraine was ranked 71st among 142 countries in the GII ranking, but as a result of fairly rapid development, it reached 46th place in 2020. However, subsequently Ukraine began to lose its positions. Thus, in 2025, it ranked 66th out of 139 countries. The Table 1 presents Ukraine's rankings relative to the Global Innovation Index and its components over the past six years [6].

Table 1

Ukraine GII Ranking: 2020–2025

Year	GII Position	Innovation Inputs	Innovation Outputs
2020	45th	74th	37th
2021	49th	76th	37th
2022	57th	75th	48th
2023	55th	78th	42th
2024	60th	78th	54th
2025	66th	80th	54th

Source: [6]

As can be seen from Table 1, in the structure of the Global Innovation Index of Ukraine, the investment component lags significantly behind the component of the effect of implementation. That is, Ukraine has a fairly high scientific potential, and increasing funding for the development of innovations and their implementation would contribute to increasing its GII. And this is one of the objectives of the national policy of innovative development of our country.

It should be noted that a comparative analysis of national innovation systems has shown [7] that the most significant differences are observed in financing mechanisms, the role of government and R&D systems. Thus, it was found that governments of developed countries provide SMEs with significant incentives, tax breaks and subsidies for the implementation of Fourth Industrial Revolution technologies, which has a positive effect on the rate of implementation. This once again demonstrates that in order to intensify the innovative development of our country, it is necessary, first of all, to increase investments in innovative activities at each stage. In this regard, we should move on to a more detailed examination of Goal number 9 from the list of Sustainable Development Goals.

Goal number 9 contains seven points [8]. In particular, paragraph 9.a provides for the following: "*Enhance scientific research, upgrade the technological capabilities of industrial sectors in all countries, in particular developing countries, including, by 2030, encouraging innovation and substantially increasing the number of research and development workers per 1 million people and public and private research and development spending*". This paragraph emphasizes that the task of government agencies in this regard is to support the development, research and implementation of innovations in the field of domestic technologies, including by creating a political climate favorable for industrial diversification and increasing added value in the raw materials industries.

Innovation as an active scientific and technological dominant of modern development and competitive activity arises as an objective reaction to the changing nature of the relationship between science and production, as well as a manifestation of their new functional relationship. For the economy of any country, it is innovation and innovative development that are the driving force that can ensure the country's economic dependence. In particular, this will allow Ukraine's national economy to bridge the gap with the global economies of developed countries based on the principle of "overtaking without catching up" (advanced development strategy). This means that the country is following the global development trend, but in its own way, seeking out and realizing its potential advantages, taking leading positions in those areas of activity where there are conditions for this. Ukraine, unfortunately, is now forced to implement this in the field of military technologies.

An objective property of innovation processes, in addition to high capital intensity and significant risks associated with their implementation (diffusion of innovations), is the uncertainty of the results, i.e., their poor predictability. This also includes a long payback period and cyclicality, that is, the wave-like nature of development, in which periods of rapid emergence of new products and their intensive implementation in various industries are followed by lulls or even stages of refinement of existing technologies. The classic idea is that any innovation goes from "birth" and rapid growth to gradual market saturation, as well as the moral obsolescence of the underlying product or technology, when something fundamentally new and modern comes to replace it. In business, this means you can't just introduce something new and call it a day. You need to prepare for the next "wave" of innovation while the current one is still profitable. In this regard, it should be emphasized once again that the transition to an innovative path of development largely depends on the position that the state takes in relation to the preservation of existing innovative potential and its increase.

Currently, state regulation of innovative development as one of the main components of the national economy has been and is being carried out in all countries of the world, including developed countries [9-14]. In modern society, innovation policy becomes a strategy for overall development, when the growth of the efficiency of social production is the result of the growth of scientific potential, its accumulation in the form of intellectual capital and the direct implementation of scientific and technological achievements. In the process of implementing innovation policy, the state acts not simply as a controller, but as its main architect and investor, which assumes financial risks in those projects that are unprofitable for private business.

The role of the state in ensuring innovative development is reduced to several key functions. First of all, this includes the development of laws, that is, the development of “rules of the game” that participants in the process must adhere to. In particular, the law should ensure the protection of intellectual property rights, as well as provide the opportunity to receive tax benefits in the early stages of implementing innovative technologies. The task of the state in this regard is to create conditions so that the introduction of innovation would be more economically profitable than working using old technologies.

Secondly, the state should take on the funding of fundamental science, which is aimed at obtaining new knowledge about the basic laws of nature, society and thinking. Although such research does not have a specific goal in the form of practical use of its results, it is they who form a system of knowledge that will become the theoretical basis for the creation of future technologies based on the latest principles. Accordingly, the state takes on the funding of fundamental research, on the basis of which companies will subsequently create new commercial products. The need for government funding of fundamental research stems from the fact that private businesses rarely invest in research that will pay off in 20 years. Although there are exceptions to this rule. Large businesses fund fundamental research when it is critical to their long-term competitiveness, even if the direct commercial benefits of that research may not become apparent for years to come. An example of such funding of fundamental research is IBM, the world's largest producer of software and consulting services. It is widely known for its commercial developments, but along with this, the company has a network of research laboratories around the world. It was for fundamental developments in the field of physics that five IBM employees became Nobel Prize laureates in different years. Thus, while working at the IBM laboratory in Zurich, Heinrich Rohrer, together with Gerd Binnig, received the Nobel Prize in 1986 for the invention of the scanning tunneling microscope [15]. And in 1990, Harry Markowitz received the Nobel Prize in Economics for their pioneering work in the theory of financial economics, namely for developing the foundations of the modern theory of portfolio choice [16]. Another example of funding fundamental research conducted by large businesses is the creation of specialized funds and institutes that analyze innovative projects and create conditions for their implementation. Thus, the Chan Zuckerberg Foundation (CZI), created by Mark Zuckerberg and Priscilla Chan, is investing billions of dollars in the Biohub project [17]. The aim of this project is fundamental research of all types of human cells, which is aimed at studying diseases and their prevention. The Foundation works closely with Stanford University, the University of California, San Francisco, and the University of California, Berkeley.

The third key function of the state in promoting innovative research is the development of infrastructure, namely, the creation of business incubators, the formation of technology parks, and even science cities, where startups can gain access to equipment and expertise. The innovation infrastructure is a multi-level support system for startups, each element of which is designed to address the challenges of a specific stage of an innovative project's development. Business incubators are focused on supporting startups at the earliest stage, which begins with the formation of an idea and ends with the creation of a prototype (Minimum Viable Product, or MVP), which can be used to test key business hypotheses. An incubator is usually understood as an organizational structure within which favorable conditions are created for the formation and rapid development of a company focused on innovation (most often, such a company is a newly created enterprise). The main goal of business incubators is to help a startup survive during the first 2–3 years. Business incubators provide their clients with production facilities (so-called physical incubators), equipped workspaces, office equipment, and also conduct the necessary business training, help in drawing up a business plan, and provide legal and accounting support.

According to statistical estimates, the global business incubator market size was approximately US$7.15 billion in 2026 and is projected to reach US$20.39 billion by 2035 [18]. The average annual growth rate of capital investment in the creation of business incubators is expected to be around 14% between 2026 and 2035. As of 2024, the global startup incubator market included over 12,000 active incubators. Of these, approximately 68% of all incubators are affiliated with universities, 22% are private, and 10% are public. Currently, business incubators provide various types of support to more than 1.5 million startups in more than 85 countries annually. In particular, there are currently more than 2,000 incubators in the United States, which accounts for almost 28% of the entire global incubation infrastructure. Of these, approximately 65% are focused on technology startups, while 18% are focused on healthcare and biotechnology startups. And such a distribution of spheres of activity is characteristic of the entire global startup market. Significant changes in the functioning of business incubators have occurred thanks to the development of digital infrastructure. As of 2024, more than 66% of all incubators operate in a hybrid or fully virtual mode, enabling expanded international collaboration. This means that startups can enter markets in three to five countries during the incubation period.

There are currently over 70 registered business incubators in Ukraine, but only about 10 are active. An example of effective assistance to startups is the Sikorsky Challenge based at National Technical University of Ukraine "Igor Sikorsky Kyiv Polytechnic Institute". This is one of the oldest ecosystems in Ukraine, combining an incubator, an accelerator, and an annual competition for innovative projects [19]. It is also worth noting Ukrainian Future (UF Incubator), which is the first full-cycle state incubator in Ukraine at the Junior Academy of Sciences and has its own prototyping laboratory, FabLab [20].

In 2024, the Cabinet of Ministers of Ukraine approved the Strategy for the Digital Development of Innovative Activity until 2030 (WINWIN), aimed at creating "centers of excellence," deregulating business, and supporting priority industries such as DefenseTech and AI [21]. A "Win-Win" strategy means there are only winners and no losers. It involves expanding access to new markets in defense, healthcare, artificial intelligence, and environmental technologies, focusing on digital transformation. One of the directions of this strategy is European integration in the field of innovation. In accordance with the European Commission's Horizon4Ukraine initiative, it is planned to involve Ukrainian researchers and specialists in projects that are already being implemented within the framework of the European Union's Horizon Europe Programme (this is one of the elements of the long-term Multiannual Financial Framework).

For more mature companies that are ready to carry out small-scale production and commercialization of technologies, more suitable innovative structures are technology parks, which are often created on the basis of universities or research institutes. Such structures can provide companies focused on innovation with laboratories for conducting research and shared-use centers with specialized equipment (industrial-grade 3D printers, physical instruments, chemical analyzers, etc.). They also provide assistance in product certification, ensure cooperation with research specialists of university, and help in finding major industrial partners. Science cities are entire districts or even cities with a high concentration of scientific potential. It should be noted that in this case, innovation plays the role of a city-forming factor. Essentially, the science city is a combination of world-class laboratories, universities and a residential environment. In science cities, startups gain access to unique facilities (reactors, synchrotrons, powerful electron microscopes, X-ray systems) and a community of world-renowned scientists. Thus, in Kharkov, with the support of the National Academy of Sciences of Ukraine, the Ministry of Education and Science, and the Kharkov City Council, one of the largest organizations of this profile, Naukograd-Kharkiv, operates, the president of which is Academician of the National Academy of Sciences of Ukraine Yuriy Matsevity [22]. The main areas of activity of the Science Park "Naukograd-Kharkiv" are: the integration of science, education and production, intensification of the use of the results of research and development work, ensuring their innovative focus, increasing the efficiency of the development, production and sale of innovative products in domestic and foreign markets, coordination of scientific, innovative, production and commercial activities of partners in scientific, technical and innovative activities, as well as assistance in attracting domestic and foreign investment.

There are many examples of the rise of entire areas of the economy as a result of improving state policy in the field of innovation. One such example is the Small Business Innovation Research (SBIR) program, also known as America's Seed Fund. This program was established in 1982 by the Small Business Innovation Act [23]. This program provides for mandatory funding, the source of which is agencies with an external R&D budget of more than $100 million per year. These agencies are required to allocate a fixed percentage of their funds (currently 3.2%) exclusively to small business financing. Eleven federal agencies participated in the program, including the Department of Defense (DoD), the Department of Energy (DoE), NASA, the National Institutes of Health (NIH), and the National Science Foundation (NSF). The implementation of this program involves three phases of interaction. The first phase involves proving the viability of the concept and its relevance. This is provided for the allocation of a grant of up to $250,000–$275,000 for a period of 6–12 months. The second phase involves the development of a prototype. A grant of up to $1.25–$1.8 million will be allocated for this purpose for a period of up to 24 months. The third and final phase is the commercialization of innovations, that is, the launch of innovative products or technologies onto the market. At this stage, the program does not provide direct funding for the project.

For 2024-2026, statistics from the Small Business Innovation Research program (SBIR program) in the United States show that this environment is highly competitive and the main barrier for companies is the justification of the innovative concept that the company puts forward, that is, the first phase. At this stage, the average success rate is only 15–-25%. When moving to the second phase, the probability of success increases significantly, and this figure already ranges from 30% to 60%. This is due to the fact that only those companies that have completed the first phase, having successfully proven the feasibility of implementing their innovative concept, are allowed to move on to prototype development. Regular studies conducted by the US Department of Defense, the National Institutes of Health and independent US think tanks demonstrate the high cost-effectiveness of the SBIR program. Thus, the figure of $22 per dollar invested is often cited in the context of analyzing all the results of the US Department of Defense over a long period. A similar figure is cited in official documents submitted to the US Senate. According to these data, the return on investment for the SBIR program ranges from $22 to $33 in economic benefits for every dollar spent [24].

Based on global experience, Ukraine's innovation policy should be formed primarily on the basis of the priority of technological development and be focused on the creation of an intellectual and information structure for the implementation of innovative design, as well as provide for the creation of opportunities for the development of innovative management, which would allow for the most effective use of the priority component of human potential in this context - intellectual potential. In this regard, a restructuring of the scientific sphere is underway in Ukraine with the allocation of an innovation sector and a search for the most effective theoretical research projects that are already at the completion stage and can be transformed into innovative projects. We are talking about creating an innovation system in our country, which is a set of public and private institutions and mechanisms that interact to create, develop and disseminate innovative technologies, knowledge and products. Such a system unites science, education, business, and the state to commercialize innovations, increasing the efficiency of the national economy. The concept of an innovation system itself was proposed by the English economist, a representative of the neo-Schumpeterian school of economic science, Christopher Freeman, back in 1982 [25], and it became widespread thanks to the works [26-28].

Based on fundamental research in the field of innovative development and structural transformations of the national economy, the following proposals were formulated for our country [29]. For the successful development of the Ukrainian economy on an innovative basis, it is necessary to improve and develop the existing elements of a modern innovative infrastructure, in particular, state development banks, technology transfer centers, scientific and industrial parks, sovereign and venture funds. These institutions are already functioning in our country, but their activities are not sufficiently effective, which is due to imperfect legislation and the lack of real incentives for innovative activity. In this regard, the goal of the state policy of Ukraine in the field of development of the innovation structure should be to change the balance of power in the economy in favor of centers of economic growth, and the basis for this should be the stimulation of investments in innovative and technological modernization and diversification of production. In addition, it is advisable to use policy instruments that provide for financial partnerships between the state and business, contribute to strengthening the country's scientific and research potential, as well as improving the quality of human potential and creating conditions for its transformation into human capital. This, in turn, requires the provision of high-quality training for specialists who are capable of using innovative technologies in their activities, thereby ensuring the diffusion of innovations. Moreover, the optimization of innovation management institutions will be a decisive contribution to the modernization of the national economy and one of the key elements in the effective use of our country's potential. It is also necessary to make fundamental changes to the existing system of statistics on innovation and scientific and technical activities in accordance with updated international standards. Modernization of monitoring of innovation activity and structural changes in Ukraine, which would involve combining the capabilities of Ukrainian databases and databases of developed countries, will accelerate European integration processes and, in the context of military and post-war challenges, will help to formulate an optimal national policy regarding science and innovation.

Today, the viability of individual enterprises and industries, as well as the competitiveness of the entire national economy in the global economic system, depends on the choice of the optimal strategy for realizing the innovative potential both at the state level and at the level of individual enterprises, on the quality of the organization of the innovation infrastructure, on the development of incentives for the use of innovation in economic activity, and on overcoming the crisis associated with the destruction of objects of national economic significance as a result of military aggression against Ukraine. The result of the successful implementation of this policy will be an increase in the public welfare of the country's population. However, the development of an optimal strategy must be based on robust scientific research, including the construction of mathematical models for the diffusion of innovations. The use of such models allows us to predict the likely socioeconomic effects and select the most effective areas of innovation development with minimal risks and costs.

2. Modeling the process of innovation diffusion

Despite the fact that the diffusion of innovations has its own specifics depending on the field of activity, on the conditions at a particular enterprise where this process takes place, its implementation has some general patterns that must be taken into account to build an adequate mathematical model of this process and subsequently when using this model to form a strategy for managing innovative development. A method that allows us to describe the general patterns of the process of diffusion of innovations and to anticipate problems that may arise during the implementation of this process is mathematical modeling based on the principle of nonlinear dynamics. At the same time, the construction of mathematical models of the diffusion of innovations involves the study of cause-and-effect relationships between the factors that determine this process, which makes it possible not only to calculate the rate of diffusion of innovations, but also to detect critical values of parameters, upon reaching which qualitative changes in the state of the economic system occur. Let's consider the general principles that need to be taken into account when constructing such models.

Depending on the formulation of the problem, modern research uses several key approaches to modeling innovation dynamics. Among them, the following groups of models can be distinguished. First of all, these are logistic models, which use a logistic (S-shaped) curve to describe the life cycle of an innovation. Models of this type were historically the first. Such mathematical models describe the process of diffusion of innovations, based on the fact that the rate of growth of the market share occupied by an innovative product first accelerates (market entry), and then slows down due to the limited capacity of the market (reaching saturation). This type of model includes the classic model of Everett Rogers, who is the author of the term “diffusion of innovations” and the founder of the theory of the same name. Although, as noted above, such models were among the first to be developed, they are still used in marketing research.

Another group of models is based on the use of matrix methods. Such models are a powerful analytical and illustrative tool in the context of state innovation policy, as they allow for the visual demonstration of complex macroeconomic indicators and the comparison of stand-ups at various levels.

The most interesting and promising area of mathematical modeling used in the study of innovation dynamics is associated with the use of differential equations to describe changes in supply and demand for innovative products in continuous time. But they are also the most difficult to construct and interpret. The undoubted advantage of such models is that they allow us not only to consider the evolution of an economic system as a transition from one state of equilibrium (unstable) to another (stable) as a result of the diffusion of innovations, but also with the help of these models it is possible to identify regimes in which bifurcations may occur, i.e. when the system is in a state of critical instability and a minor change in external conditions (taxes, investment volume or level of education) leads to abrupt and irreversible changes in the path of economic development. One of the first models of this type is the model of Frank Bass, who “translated” Rogers’ logistic model into the language of mathematics, using the apparatus of differential and integral calculus. It should be noted that to construct similar mathematical models of dynamics, but in discrete time mode, the mathematical apparatus of difference equations is used.

Finally, another effective area of research into the diffusion of innovations is simulation modeling. Its implementation uses ready-made mathematical constructs, on the basis of which, with the help of software, various scenarios for the development of the innovative potential of an enterprise (or region) are reproduced, depending on the influence of such factors as the volume of investments, certain risks, the presence of competition in the innovation market, and the like. Quite often, simulation modeling involves testing "What if?" scenarios. Since the process of innovation diffusion is nonlinear, simulation modeling is an effective method for identifying so-called "inflection points" when a qualitative change in the behavior of an economic system occurs. For example, while an innovation circulates on the periphery among disparate groups of similarly intended products, the growth in the number of new users (or new buyers) is linear. However, once the virus of a new idea reaches highly connected opinion leaders, the simulation model can detect a sharp exponential rise in the number of users. The advantages of simulation models include their ability to visualize complex processes described within dynamic models, such as the transition of an economic system into a state of self-oscillation or chaos.

Let us consider some examples of classical mathematical models of innovation dynamics. As noted above, one of the founders of the theory of diffusion of innovations is considered to be the American sociologist and communication theorist Everett Rogers. He defined its main objectives as the study of social and managerial factors that determine the dynamics of the dissemination of innovative ideas, innovative technologies, and innovative products. In accordance with the theory of diffusion of innovations proposed by Rogers in 1962 and which later became a classic [30], it is assumed that the spread of innovations occurs through the transmission of information, which may take the form of, for example, a mathematical model, to members of a social group through various channels. The speed and effectiveness of this process depend on the extent to which this innovation corresponds to the values of the social group accepting it. Diffusion of innovations is a sequential change of four stages. At the first stage, the audience needs to be informed about the content of the innovation (*knowledge about innovation*). The second stage requires convincing the audience or social group that they need this innovation (*persuasion*). In the third stage, the audience must make a decision to purchase and/or receive the innovation (*decision*). The fourth stage assumes that after receiving the innovation, the social group must constantly confirm the correctness of its own decision about the usefulness of the received innovation (*confirmation*). Based on the S-shaped curve, Rogers [30] constructed a diagram reflecting the diffusion of innovations (Fig. 1).

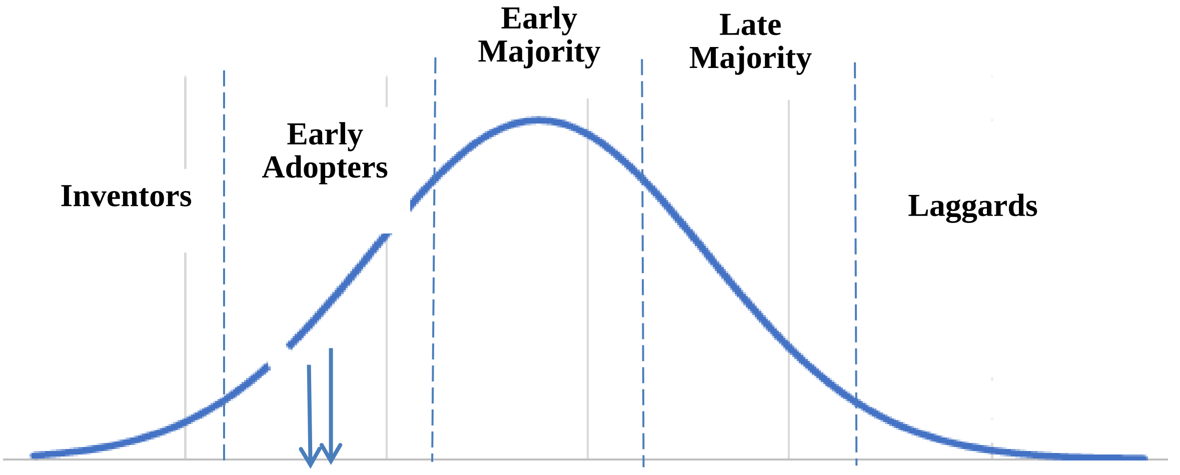

Figure 1. Changes in the number of buyers of innovative products over time (Rogers's curve)

Source: [30]

In examining the change in the number of buyers of innovative products over time, Rogers took into account the distribution of participants in the process of diffusion of innovations into groups depending on their attitude towards innovations of any type. As Figure 1 shows, Rogers identified five social groups involved in promoting innovation. This division is based on a normal distribution curve (Gaussian function). Rogers characterized these groups as follows. The first group, based on the timing of innovative product acquisition, consists of innovators who possess significant financial resources and technical knowledge and are willing to take risks by purchasing new products that are still little known on the market. Inventors constitute approximately 2.5% of the total number of buyers of this product. The second group is early adopters. These are people whose opinions are listened to and who are sought for advice. They make up 13.5% of the market capacity. They are quick to adopt new things, but more cautious than inventors. The third group is the early majority. Their market share is approximately 34%. These are pragmatic people who accept an innovation only after they receive evidence of its benefits from early adopters. The fourth group includes skeptics, or the so-called late majority (34%). They start using a new product or technology only when it becomes a generally accepted standard and it is no longer convenient not to use it. The last group, the remaining 16%, are conservatives who look to the past and accept something new only when what was new has become commonplace and even begun to become outdated.

One more feature that should be noted is shown in Figure 1. Rogers emphasized that the most difficult and risky stage is the dissemination of innovations among early adopters. Most often, it is at this stage that an innovation can be rejected by the social environment, therefore, within the stratum of early adopters, Rogers identifies a gap that he called the "Chasm" (it is indicated by arrows), that is, the place where problems with the "survival" of innovations most often arise.

Today, the innovation diffusion model proposed by Everett Rogers is considered the foundation of modern marketing, management and sociology. It should be noted that although this model has received wide acceptance, its application is limited. In essence, it is a communication model and rather states than explores cause-and-effect relationships. In particular, in marketing, Rogers' model is used primarily to denote product life cycles.

In 1969, American scientist and researcher in the field of marketing Frank Bass, based on the work of Everett Rogers, proposed a mathematical model for the distribution of innovative products [31], using the mathematical apparatus of differential and integral calculus for its construction. But unlike the model proposed by Rogers, Bass identified only two social groups involved in the process of innovations diffusion. The first group (innovators), perceiving information about the innovation from advertising, become its first buyers. This is the so-called "advertising effect". The second group (imitators) begins to use the innovative product after receiving information about it from interpersonal communication. This effect is called "word of mouth". The Bass mathematical model is based on the assumption that the purchase of a new product at any given time is a linear function of the number of consumers who have already purchased this product. His proposed model uses the same approach as epidemic models. Bass drew an analogy between the spread of information about a new product and the spread of a virus. The purchase decision is considered as a probability that depends on the current point in time and the proportion of those who have already implemented the innovation. Based on this, Bass constructed a mathematical model describing the dynamics of sales of innovative products over time.

The Bass model is a first-order differential equation that describes the rate of adoption of an innovation at a point in time *t*:

$$\frac{dN(t)}{dt} = \left(p + q\frac{N(t)}{m} \right) \cdot \left(m - N(t) \right), \tag{1}$$

where $N(t)$ is the cumulative (total) number of consumers who have already purchased an innovative product by a given point in time t; m is potential market capacity ("ceiling"), i.e. the total number of people who can buy this innovative product; p is the innovation coefficient, which shows how often people buy a new product under the influence of advertising, that is, independently of each other; q is the imitation coefficient, which shows how much the opinions of those buyers who have already purchased an innovative product influence the opinions of future buyers (the word-of-mouth effect).

As a result of integrating equation (1), we obtain its solution in the following form:

$$N(t) = m\frac{p - e^{-(p+q)t}}{p + q \cdot e^{-(p+q)t}}. \qquad (2)$$

In fact, the coefficients p and q show the ratio of innovators and imitators in the regional system in which the diffusion of innovation occurs. The coefficient p determines the share of innovators at the initial stage of the innovation diffusion process. At the initial moment of time, the value of the function $N(t)$ is close to zero, and the main sales occur due to advertising. Over time, the share of innovators gradually decreases, but as a result of growth of the function $N(t)$, the share of imitators increases, the number of which is directly proportional to the number of existing consumers. This leads to an exponential growth in the number of consumers of innovation. If $N(t) \to m$, i.e., when the market tends to be saturated, there is a slowdown in the growth of the total number of buyers of innovative products. Thus, the change in the total volume of sales of innovative products over time is described by an S-shaped curve.

Depending on the values of the coefficients p and q, two limiting cases may exist. If $p > 0$ and $q = 0$, the graph of function (2) is an exponential. This model is called the “external influence model.” If $p = 0$ an $q > 0$, the graph of function (2) is a classical logistic curve, and the model is called the “internal influence model”. This model is also known as the Mansfield's Model [32]. In this case, the differential equation (1) is simplified to the classical logistic form:

$$\frac{dN(t)}{dt} = \frac{q}{m} N(t)\left(m - N(t)\right). \qquad (3)$$

According to equation (3), the rate of adoption of an innovation is directly proportional to the number of consumers who have already purchased the innovative product $N(t)$ and the number of those who can purchase it $(m - N(t))$.

The general solution of equation (3) is a function that describes the logistic curve:

$$N(t) = \frac{m}{1 + C \cdot e^{-qt}}, \qquad (4)$$

where C is constant of integration, which is determined by the initial conditions. If at the initial moment of time t_0 the number of adopters is N_0, then the constant is equal to $C = \frac{m - N_0}{N_0}$.

In general, the mathematical model proposed by Bass allows us to describe the diffusion of innovative products depending on various factors that influence management decision-making.

In a joint work by Bass, Krishnan and Jain [33], a generalized mathematical model of innovation diffusion was proposed, which, in addition to variables such as product price and advertising, which directly affect the rate of introduction of innovative products and technologies, also includes an influence coefficient that reflects current marketing activity. In this interpretation the model takes the form:

$$\frac{dN(t)}{dt} = \left(p \cdot m + (q - p) \cdot N(t) - \frac{q}{m} N^2(t) \right) \cdot x(t), \quad (5)$$

where $x(t)$ is influence coefficient, which is a characteristic of current marketing activity, or the marketing effort function.

The marketing effort function is described by the equation:

$$x(t) = 1 + a \frac{P'(t)}{P(t)} + b \frac{A'(t)}{A(t)}, \quad (6)$$

where a and b are coefficients corresponding to the weight (significance) of the price of innovative products and the costs of their advertising, respectively; $dP'(t) / dP(t)$ is relative change in price at a point in time *t*; $dA'(t) / dA(t)$ is relative change in advertising costs at a point in time t.

It should be emphasized that the indicator $N(t)$ is cumulative and reflects the number of adopters (buyers) who at a specific point in time t have already purchased innovative products, while the indicator $n(t) = dN(t) / dt$ reflects the number of new adopters strictly at a specific point in time t.

A study of the influence of the parameters of equation (5) on the form of the function describing changes in the number of new adopters over time shows that, depending on the ratio of the parameters p and q, a shift in the curve described by Rogers can be observed, and in the limiting case, when the growth in the number of adopters occurs only due to innovators ($p > 0$; $q = 0$), then a change in the shape of the curve also occurs (Fig. 2)

Currently, the concept proposed by Bass, which involves the use of the mathematical apparatus of differential and integral calculus to describe and forecast the processes of innovations diffusion, has found its embodiment in a large number of mathematical models of innovation dynamics.

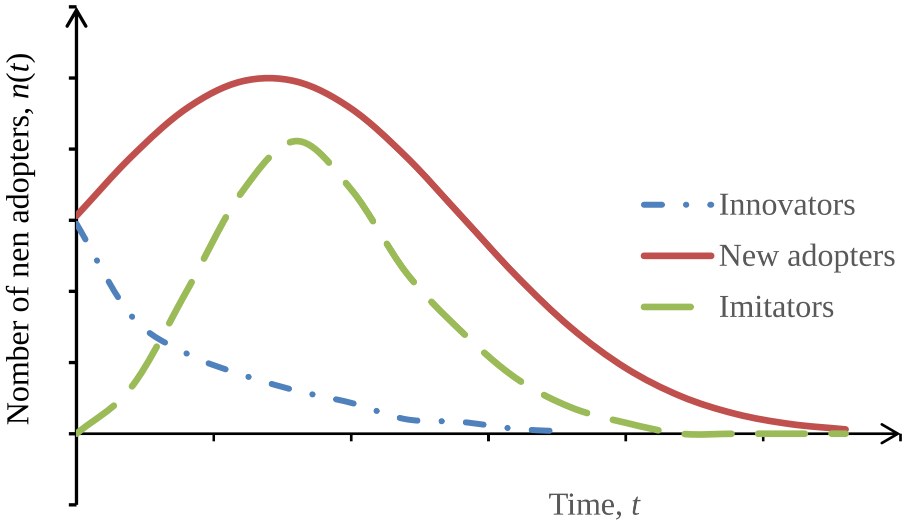

Figure 2. The influence of parameters *p* and *q* on the dynamics of change in the number of new adopters in accordance with the generalized Bass model

Source: generalized by the authors to bases [31; 32]

If we take into account that the dynamics of innovation diffusion is a nonlinear process, as well as the possibility of the economic system transitioning to a chaotic state, then it makes sense to consider models in which the diffusion process ceases to be viewed as an S-shaped curve and begins to exhibit more complex regimes. Such models include, for example, the innovation competition model.

Both the Rogers model and the Bass model described above consider the diffusion process of a single innovative product. However, in reality, two innovative products with similar areas of application may be present on the market at the same time. In this case, competition arises between them. In addition, an example of competition in the innovation market can be the behavior of large corporations, for whom it is more profitable to absorb a smaller innovative company than to invest in scientific research themselves. To construct mathematical models of such processes in economics, an approach is used that was proposed at one time by the American mathematician and physical chemist Alfred J. Lotka and the Italian mathematician and physicist Vito Volterra to describe competition in a biological environment. These scientists considered the coexistence of two populations in an isolated environment, which provides everything necessary only to one population, and individuals of the second population feed on individuals of the first population. This model was named "Predator-Prey". It describes how the numbers of each of the two biological species that depend on each other change.

The mathematical model of the Lotka-Volterra problem is a system of two ordinary differential equations, the solution of which reveals the presence of periodic fluctuations in the numbers of predators and prey. The reason for this cyclical nature is as follows. If the prey population is large, predators begin to reproduce rapidly because they have plenty of food. But when predators become numerous, they actively consume their prey, causing the prey population to decline sharply. Consequently, the predators have nothing to eat, and their numbers decline. And as predators become scarce, the prey population begins to increase. Thus, the cycle repeats itself.

The Lotka-Volterra model not only became the foundation of mathematical biology, but also found its application in describing the behavioral characteristics of economic systems, helping to determine the limits of their stability and to study the characteristics of the states into which the system can transition. In particular, the predator-prey approach is used to analyze the interaction of two economic agents, where one agent (the predator) develops at the expense of the resources of the other (the prey), which leads to cyclical changes in their states, for example: producers and consumers, small and large businesses, innovations and obsolete technologies or, as mentioned above, the interaction between competing innovative products. But before moving on to the analysis of the Lotka-Volterra mathematical model of innovation competition, it is necessary to introduce into consideration such a concept as an innovation ecosystem, which is used in this model.

Initially, the concept of an ecosystem was introduced in biology as an arena for interaction between living organisms and their environment and was used as a physical model of nature when constructing the “Predator-Prey” model. In relation to innovation and competition in the economy, this concept was introduced by the American economist James F. Moore [34], who is considered the founder of the ecological approach to competition. He proposed viewing a company not as a separate player in an industry, but as part of a “business ecosystem” whose participants co-evolve, compete, and create value. Moore views a business ecosystem from the perspective of its life cycle, which he divided into four successive stages: birth, expansion, leadership, and self-renewal or death. At the birth stage, the leader creates a product that is superior to existing analogues and, working with partners, forms a team of allies. As a result, the ecosystem begins to capture new territories (expansion stage). In fact, this is a stage of aggressive marketing. At the leadership stage, the leader, as the center of the ecosystem, must create an incentive for partners to continue developing within the systems, which he does through the redistribution of income. When faced with threats from new technologies, ecosystems must either renew themselves through the transition to innovative products or disintegrate. As a participant in the ecosystem, a company is required to perform certain actions. One group of actions is aimed at forming and maintaining cooperative relationships with business partners, and the second is aimed at competitive struggle.

In the Lotka-Volterra model, when applied to the description of the process of diffusion of innovations, the ecosystem is treated as a dynamic market niche in which different technologies or companies interact like biological species. In this mathematical model of nonlinear dynamics, the ecosystem is represented as a system of mathematical dependencies that take into account that the survival of one innovation directly depends on the behavior of another. This is accomplished in the following way. First, the ecosystem determines the resource base, i.e., the volume of “food” for the predator (market potential, demand capacity), and also takes into account resource limitations, which forces innovations to compete for the same consumer.

Secondly, the nature of the interaction between innovations is specified. This can be competition, when an innovation acts as a predator and absorbs the market share of an old technology or another innovation that is similar in focus (for example, the market for smartphones of various brands or the market for various artificial intelligence chats). However, collaboration between innovations can also be observed (for example, the computer market involves parallel improvement of both hardware and software). It is also possible for one innovative technology to parasitize on another (for example, the AdBlock blocker program parasitizes on various platforms such as YouTube).

Third, it is assumed that the ecosystem has limited market capacity. Accordingly, the number of users of innovative products cannot grow indefinitely. It either reaches saturation (stabilizes) when the market limit is reached, or the innovation dies if it is displaced by a predator more adapted to the given market. In an innovation ecosystem, there is not a movement of energy, but a movement of capital and other economic resources. Economic resources here include not only material resources (monetary funds, equipment, movable and immovable objects, etc.), but also human capital. Accordingly, one of the key factors that must be taken into account when constructing a mathematical model of an ecosystem is the interaction coefficient, which shows how much the emergence of one innovative product slows down or accelerates the spread of another innovative product.

A classic example of the use of the Lotka-Volterra model in relation to the analysis of the diffusion of technological innovations and their competition is considered to be the work of the South African economist Carl W.I. Pistorius and the American economist James M. Utterback [35]. While the interaction between technologies is usually understood as confrontation, the authors consider this issue in a broader sense than just pure competition. To assess the interaction of two or more technologies, they propose using a multi-mode model of behavior (pure competition, symbiosis, and predator-prey interaction), also taking into account that the interaction between technologies can generally temporarily switch from one mode to another.

Using the same notation as in the Rogers and Bass models, the Lotka-Volterra mathematical model in the context of innovation dynamics, which is a system of two differential equations, can be written as follows:

$$\begin{cases} \dfrac{dN_1(t)}{dt} = \left(p_1 + q_1 \dfrac{N_1(t)}{m} \right) \left(m - N_1(t) - \alpha_{12} N_2(t) \right); \\ \dfrac{dN_2(t)}{dt} = \left(p_2 + q_2 \dfrac{N_2(t)}{m} \right) \left(m - N_2(t) - \alpha_{21} N_1(t) \right), \end{cases} \tag{7}$$

where $N_1(t)$ and $N_2(t)$ are the cumulative (total) number of consumers who have already purchased the first or second innovative product, respectively, by the time *t*; p_i is the rate at which innovators adopt a particular the *i*-th new technology ($i = \overline{1, 2}$) independently of others; q_i is the rate of diffusion of the *i*-th technology ($i = \overline{1, 2}$) as a result of an increase in the number of adopters (network effect) taking into account the pressure of competing technology; α_{12} and α_{21} are competitive influence coefficients (degree of technology interchangeability).

As can be seen from the equations of system (7), the expression in the first brackets of the right-hand side of each differential equation characterizes the growth rates of each innovation separately and depends on the investment attractiveness of the given innovation. The expression in the second brackets characterizes the residual potential of the market. To determine it, instead of the multiplier $\left(m - N(t) \right)$, the multiplier $\left(m - N_i(t) - \alpha_{ij} N_j(t) \right)$ is used, since the available market for the *i*-th innovative technology is reduced not only by its own sales, but also by the sales of the *j*-th (competitive) technology, multiplied by the influence coefficient α_{ij}

The influence coefficient α_{ij} shows what proportion of potential buyers the *i*-th innovative technology loses due to sales of the *j*-th technology. If $\alpha_{ij} = 1$, then both types of innovative products are completely interchangeable. In the limiting case, when $\alpha_{ij} = 0$, i.e., there is neither competition nor symbiosis between the diffusion of different innovative products, we obtain for each of the innovations its own Bass model. Thus, the economic system described by the mathematical model (7) is unstable in the sense that one of the participants can have either an infinitely large profit or zero, depending on the value of the coefficient α_{ij}. A graphical illustration of innovation competition according to the Lotka-Volterra model is shown in Fig. 3.

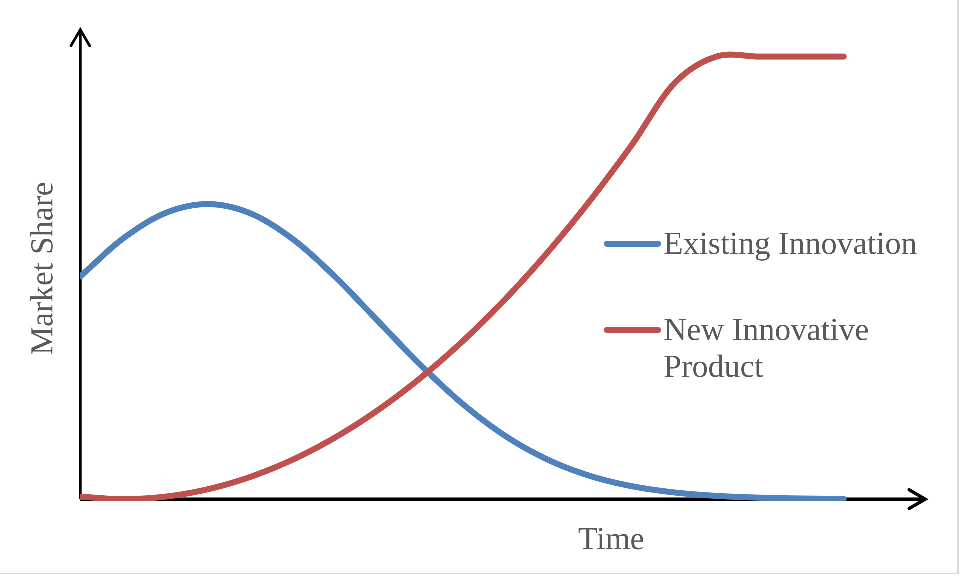

Figure 3. Competition of innovative products according to the "Predator-Prey" model

Source: generalized by the authors to bases [35]

As can be seen from the comparison of mathematical models of innovation diffusion in the Bass and Lotka-Volterra representations, these models are based on some common patterns. In particular, both of them assume that the growth of the total number of adopters is described by a logistic *S*-shaped curve. In this regard, the transition from the Bass model to the Lotka-Volterra model can be viewed as a transition from monopoly to competition, and the effect of displacement of one innovative product by another in the Lotka-Volterra model is, in essence, the negative impact of imitators of a competing innovative product on the potential market. However, it should be noted that the Lotka-Volterra model, presented in the form of a system of differential equations (7), does not take into account the influence coefficient *x(t)*, which is a characteristic of the current marketing activity. There are many works that provide a comparative analysis of these mathematical models of innovation diffusion and propose ways for their further development. In particular, the classical Lotka-Volterra model of innovation diffusion assumes that both competing innovations appear on the market simultaneously (synchronous competition), whereas this is an approximation. With sequential entry into the market, the emergence of a new innovative product leads to the absorption of the potential market share (diachronic competition) and disrupts the distribution of the product that already exists on the market. In accordance with this, in the work [36] the authors propose a model that allows for a change in the values of the diffusion parameters of an innovative product as soon as another innovation appears on the market.

Currently, mathematical models of innovation competition using the “Predator-Prey” relationship model are actively used to analyze dynamic effects observed in the innovation market, as well as to analyze possible development scenarios and select the optimal strategy. As an example [37], we can cite the application of the Lotka-Volterra model to the analysis of such a pressing problem as the evolution of market shares of renewable energy and electricity from renewable sources. Another interesting example [38] is the study of the specifics of using various mathematical models of innovation diffusion to analyze the distribution of smartphones of competing operating systems iOS and Android at two levels: as a brand and as a product.

It is interesting to note [39] that the current rate of spread of information technologies, especially artificial intelligence and mobile applications, exceeds classical benchmarks by 2.34 times**.** Analytical research attribute this to the fact that the share of imitators among adopters in the innovation market has begun to significantly exceed the share of innovators. The development of social media has led to the word-of-mouth algorithm becoming the primary driver of sales, making the innovation market difficult to predict. And due to the high interaction rate on social media, the number of new adopters reaches its maximum very quickly.

3. Cyclicality of innovation dynamics

Let us move on to an analysis of the specifics of mathematical models of dynamics used to describe the diffusion of innovations. Dynamic processes can be described using both linear and nonlinear mathematical models. Linear models are the simplest to construct and interpret, and although they have a limited scope of application, they are the most popular, especially in the initial stages of modeling. Linear models of dynamics are based on the assumption that the predictors, that is, the parameters used to construct the model, retain constant values within the time period for which the model is created, and the dependence of the endogenous (internal) factor on the exogenous (external) factor is described by a linear function. As noted above, the Bass model is based on the assumption that the diffusion of an innovative product is a linear function of the number of consumers who have already purchased this product, i.e., the model he proposed is linear. However, linear models do not take into account either random factors that may be present in the ecosystem or uncertainties, and this can significantly reduce their accuracy even for short-term forecasts, while such models are generally of little use for long-term forecasts. Therefore, in the case where forecasting needs to be carried out for a sufficiently long period of time, as well as for a more adequate reflection of the processes occurring in the economic system, it is advisable to use nonlinear dynamic models. It is obvious that forecasting based on such models can provide greater forecast accuracy over a short period of time.

Unlike linear models, which are based on the assumption that within the period of operation of a system, its future state is determined by a simple superposition of the influence of various factors, the numerical characteristics of which are unchanged over time, nonlinear models take into account the non-additivity of the results of internal processes occurring in the system. Such models provide for the possibility of the emergence of synergistic effects, when the result of the simultaneous influence of factors due to their interaction exceeds the sum of the results of the influence of each factor separately. Accordingly, nonlinear models are capable of foreseeing and describing such phenomena as loss of equilibrium, occurrence of limit cycles, bifurcations and transition of the system into a state of chaos. The construction of mathematical models based on nonlinear dynamics is of exceptional importance in the study of innovation processes, since the innovative development of an economic system is nonlinear in nature. It should be noted that another feature of nonlinear models is the existence of a threshold. If the number of adopters of an innovative product is below a certain level, then the innovation is not viable. If, on the contrary, in the process of rapid distribution it has occupied a critical market share, an explosive, avalanche-like growth in the number of users occurs. Both of these effects are described using a nonlinear feedback function.

The nonlinearity of the model is also manifested in the fact that its parameters should be considered time-dependent, and this is also taken into account using the feedback function. Moreover, the direction of feedback usually changes during the process of dissemination of an innovative product. Positive feedback usually corresponds to the initial stage of innovation diffusion. This is manifested in the fact that the more people use a new product or new technology, the wider the information about it spreads (network effect), and this further accelerates the diffusion process. As the market becomes saturated, the feedback loop reverses. Negative feedback manifests itself in the fact that when the market is close to saturation, the cost of acquiring a new customer increases nonlinearly, and product fatigue also occurs.

Let us now proceed directly to the construction of our own mathematical model of the diffusion of innovations, without making any particular simplifications regarding this process, that is, we will take into account that the innovative evolution of an ecosystem is a substantially nonlinear process and that abrupt transitions from one equilibrium position to another are possible, as well as the emergence of self-oscillations and a transition to a state of chaos. In this case, to justify the methods of qualitative forecasting of possible states of the economic system by phase trajectories, the following hypotheses were adopted as the main ones:

– synthesis of patterns of behavior of stationary states and cyclic dynamics during the implementation of limit cycles of different periodicities;

– the emergence of both medium-term and long-term cycles, which have significant differences in the frequency and intensity of their impact on the structure of the economic system.

It should be emphasized that the emergence of fluctuations in the ecosystem associated with the diffusion of innovations can be the cause of technological crises with accompanying changes in the structure of innovative products. The reason for the emergence of the cycle should be considered the decline in the efficiency of dominant technologies, which leads to a decrease in the rate of economic development. This in turn stimulates scientific and engineering activity towards generating new technical breakthroughs. However, it is well known that, simultaneously with the acceleration of economic development, the influence of restraining factors that negatively affect the growth of national income also increases. In this case, economic growth either stabilizes or small fluctuations occur near the equilibrium value. This statement can be illustrated using a mathematical model that takes into account the existence of a feedback loop between the rate of economic growth and the amount of aggregate national income. This feedback plays the role of a regulator in the ecosystem. It is believed that the influence of economic environment resistance increases with the cumulative growth of income. In other words, it can be assumed that the growth rate of the volume of production of innovative products depends on the average weighted total volume of production in the past, and not only on its volume at a given point in time. Accordingly, the dynamics of the innovation process can be described using an integral-differential equation:

$$\frac{dy}{dt} = y\left(L - \int_0^t K(t-\tau)\cdot y(\tau)\cdot d\tau \right), \tag{8}$$

where $y = y(t)$ is the volume of production of an innovative product, which is a time-dependent function; $K(t)$ is a function that determines the cumulative growth of the production of innovative products; τ is a point in time in the past that determines the distance from the point in time t for which the volume of production of an innovative product is determined (distributed delay, or time lag); L is a parameter that has the meaning of a technological limit to production growth (market saturation), respectively, $L > 0$.

It should be noted that another distinctive feature of nonlinear models is that they are constructed taking into account the presence of a time lag, i.e., delay. In this model, we consider a distributed lag, which is evident from equation (8), since τ is the integration variable. In classical models, the market reaction is considered instantaneous, that is, there is no time gap between “found out” and “decided”, whereas in real conditions this gap exists and can have different durations. For every real economic system there is a critical value $\tau_{cr.}$. If $\tau < \tau_{cr.}$, the distribution of innovative products is stable, however, if $\tau > \tau_{cr.}$, then cyclic instability will arise. These features need to be taken into account when conducting an advertising campaign for a new product or other promotions aimed at attracting the attention of buyers.

The most obvious example of the occurrence of a delay (time lag) in relation to innovation is the presence of strict deadlines for updating technology. This means that the rate of diffusion of an innovation depends on how many people (or firms) bought the product exactly $\tau_{cr.}$ months or years ago, and, accordingly, now it is time for them to update it. If $\tau_{cr.}$ is about 1 year (products quickly become obsolete), then diffusion occurs smoothly. The market quickly becomes saturated and then lives on a steady stream of updates. If $\tau_{cr.}$ is about 5 years (long-lived product), then an "echo" of the first wave of sales occurs. Five years after the start of sales, there will be a powerful surge that can crash servers or create a shortage, and then there will be a decline, followed by a long period of inactivity.

As a rule, a function $K(t)$ with an appropriate set of characteristic delay times is used as a control function. If we use the terminology of the theory of automatic control, then the function $K(t)$ is a pulse-transient function of a linear controller, to the input of which a signal $y(t)$ is received. The output of such a system is the value $U(t) = \int_0^t K(t-\tau) \cdot y(\tau) \cdot d\tau$, which is the convolution of two given functions. It is convenient to use the operator representation of a function $K(t)$ in the form of a fractional rational function of arbitrary order:

$$K_n(\lambda) = \frac{b_{n-1}\lambda^{n-1} + \ldots + b_1\lambda + b_0}{\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0} \tag{9}$$

with the fulfillment of the normalization condition

$$\int_0^\infty K_n(t) \cdot dt = 1,$$

from which it follows that $a_0 = b_0$.

The nonlinear integral-differential equation (8) has two equilibrium positions: $y^* = 0$ and $y^* = L$. Since we are considering the process of diffusion of innovations, the trivial equilibrium position $y^* = 0$ is not relevant for this study, since such a position is stable if there are no innovations. The equilibrium position $y^* = L$ has the meaning of a technological limit of income growth (saturation of the market with new products) and is an important indicator. Our task is to study the dynamic behavior of the process of diffusion of innovations in a small neighborhood of this equilibrium position.

Let us introduce a new variable $\tilde{y}(t) = y(t) - L$ that has the meaning of deviation from the equilibrium position $y^* = L$. Then equation (8) can be rewritten as follows:

$$\frac{d\tilde{y}}{dt} = -L\int_0^t K_n(t)\cdot\tilde{y}(t-\tau)\cdot d\tau - \tilde{y}\int_0^t K_n(t)\cdot\tilde{y}(t-\tau)\cdot d\tau\,. \tag{10}$$

The characteristic equation of the linear part of equation (10) takes the form:

$$\lambda + L\cdot K_n(\lambda) = 0, \tag{11}$$

where the function $K_n(\lambda)$ is defined by the relation (9).

When $n = 1$, we obtain that $K_1(\lambda) = \dfrac{a_0}{\lambda + a_0}$, at the same time $a_0 > 0$. Then the characteristic equation (11) is a quadratic equation with respect to the eigenvalues λ and has the form:

$$\lambda^2 + a_0\lambda + La_0 = 0\,.$$

It is obvious that the real parts of the roots of this quadratic equation are negative, and the equilibrium position $y^* = L$ can be either a stable node or a stable focus. In this case, the transition from the node to the focus is not a bifurcation, since the stability conditions are preserved.

Let us consider a more interesting situation (in terms of the diversity of dynamic behavior of the ecosystem), which can occur when $n = 2$. Let's assume that

$$K_2(\lambda) = \frac{a^2}{\lambda^2 + 2\xi a\lambda + a^2}, \quad \xi > 0\,,$$

which is a typical second-order link in automatic control theory. In this case, the characteristic equation (11) is a third-degree equation with respect to λ and is transformed to the form:

$$\lambda^3 + 2\xi a\lambda + a^2\lambda + La^2 = 0\,. \tag{12}$$

If the relation between the coefficients of the cubic equation (12) is satisfied

$$L = 2\xi a\,, \tag{13}$$

then we obtain the roots of the cubic equation in explicit form:

$$\lambda_{1,2} = \pm i \cdot a, \quad \lambda_3 = -2\xi a, \quad i^2 = -1.$$

They determine the behavior of equations (8) and (10) at the stability boundary. Considering that these integral-differential equations are nonlinear, there is a possibility of the emergence of a self-oscillation mode, i.e., the appearance of a limit cycle as a consequence of the Hopf bifurcation. In the theory of diffusion of innovations, the Andronov-Hopf bifurcation describes the effect when the stable spread of innovative products is suddenly replaced by self-oscillations. If, in accordance with the classical Bass model, an economic system moves from one state of equilibrium to another, then the occurrence of an Andronov-Hopf bifurcation leads to the emergence of a closed trajectory (limit cycle), that is, the system does not evolve, but “goes in circles”. In terms of innovation, this manifests itself in the following way: Upon reaching a certain threshold, interest in a new technology suddenly drops sharply, and then (after a price drop, for example) it rises just as sharply.

Taking into account the representation of the characteristic polynomial in the form (12), we transform the integral-differential equation (10) to a system of three ordinary differential equations. To do this, we introduce new variables: $y_1 = \tilde{y}$, $y_2 = U - L$, $y_3 = \dfrac{dU}{dt}$. As a result of the transformations, we obtain:

$$\begin{cases} \dfrac{dy_1}{dt} = -L \cdot y_2 - y_1 \cdot y_2; \\ \dfrac{dy_2}{dt} = y_3; \\ \dfrac{dy_3}{dt} = a^2 y_1 - a^2 y_2 - 2\xi a y_3. \end{cases} \tag{14}$$

Let us move on to considering Hopf's bifurcation theorem as applied to a system of differential equations (14). Let us start with the choice of a bifurcation parameter in system (14), upon reaching a critical value of which a number of imaginary eigenvalues $\lambda_{1,2}$ can exist. It is convenient to choose as a small variable parameter $\mu = L - 2\xi a$ and the frequency of oscillations

$$\omega = a \; (\mu = 0). \tag{15}$$

Then the characteristic equation (12) takes the following form:

$$\lambda^3 + 2\xi a\lambda + a^2\lambda + a^2(2\xi a + \mu) = 0. \tag{16}$$

Let us find the derivatives of the eigenvalues with respect to the parameter μ at $\mu = 0$ and $\lambda_{1,2} = \pm i \cdot a$:

$$\lambda'(0) = \frac{1}{2} \cdot \frac{1}{1+4\xi^2} + i \cdot \frac{\xi}{1+4\xi^2},$$

where

$$\operatorname{Re}\lambda'(0) = \alpha'(0) = \frac{1}{2} \cdot \frac{1}{1+4\xi^2}; \qquad \operatorname{Im}\lambda'(0) = \omega'(0) = \frac{\xi}{1+4\xi^2}.$$

Since $\alpha'(0) \neq 0$, it can be argued that the conditions of Hopf's theorem are satisfied and the birth of a limit cycle from a complex focus takes place.

The next step is to construct the Poincaré normal form for the system of differential equations (14). This is necessary to conduct a direct study of the direction of birth or death of the limit cycle, its period, amplitude and other parameters. To do this, we will make another change of variables:

$$y_1 = 2\xi \cdot x_1 + x_3; \qquad y_2 = x_2 + x_3; \qquad y_3 = a \cdot x_1 - 2\xi a \cdot x_3.$$

After transformations we obtain system (14) in new variables x_1, x_2 and x_3:

$$\begin{cases} \dfrac{dx_1}{dt} = \quad -ax_2 \qquad + F_1(x_1, x_2, x_3); \\ \dfrac{dx_2}{dt} = ax_1 \qquad\quad + F_2(x_1, x_2, x_3); \\ \dfrac{dx_3}{dt} = \qquad -2\xi a x_3 + F_3(x_1, x_2, x_3), \end{cases} \tag{17}$$

where

$$F_i(x_1, x_2, x_3) = A_i \cdot f(x_1, x_2, x_3), \quad i = \overline{1, 3},$$

$$f(x_1, x_2, x_3) = 2\xi x_1 x_2 + 2\xi x_1 x_3 + x_2 x_3 + x_3^2,$$

$$A_1 = \frac{2\xi}{1+4\xi^2}, \quad A_2 = \frac{-1}{1+4\xi^2}, \quad A_3 = \frac{1}{1+4\xi^2}.$$

To reduce the order of system (17), we will bring it into complex-valued form by introducing new coordinates:

$$z = x_1 + i \cdot x_2; \ \overline{z} = x_1 - i \cdot x_2; \quad v = x_3.$$

In this coordinate system, the system of three differential equations (17) is reduced to a system of two differential equations:

$$\begin{cases} \dfrac{z}{dt} = i - a \cdot z + G\left(z, \overline{z}, v\right); \\ \dfrac{v}{dt} = -2\xi a \cdot v + H\left(z, \overline{z}, v\right), \end{cases} \tag{18}$$

where

$$G\left(z,\overline{z},v\right)=F_1\left(z,\overline{z},v\right)+i\cdot F_2\left(z,\overline{z},v\right),$$
$$H\left(z,\overline{z},v\right)=F_3\left(z,\overline{z},v\right).$$

Using the central manifold method, we express the variable v through z and $\overline{z}$ using the relation $v=W\left(z,\overline{z}\right)$, where

$$W\left(z,\overline{z}\right)=w_{20}\frac{z^2}{2}+w_{11}z\cdot\overline{z}+0\left(\left|z\right|^3\right), \tag{19}$$

accordingly,

$$w_{20}=-\frac{1}{2}\cdot\frac{h_{20}}{\xi a+i\cdot a};\quad w_{11}=-\frac{h_{11}}{2\xi a}.$$

And we receive

$$h_{11}=\frac{1}{4}\left(\frac{\partial^2F_3}{\partial y_1^2}+\frac{\partial^2F_3}{\partial y_2^2}\right)=0\quad\Rightarrow\quad w_{11}=0;$$

$$h_{20}=\frac{1}{2}\left(-i\cdot\frac{\partial^2F_3}{\partial y_1\partial y_2}\right)=-\frac{i\cdot\xi}{1+4\xi^2}\quad\Rightarrow\quad w_{20}=-\frac{\xi+i\cdot\xi^2}{2a\left(1+4\xi^2\right)\left(1+\xi^2\right)}.$$

Substituting expression (19) into the system of equations (18), we obtain the differential equation:

$$\frac{dz}{dt}=i\cdot az+g_{20}\frac{z^2}{2}+g_{11}z\cdot\overline{z}+g_{02}\frac{\overline{z}^2}{2}+g_{21}\frac{z^2\cdot\overline{z}}{2}+\ldots, \tag{20}$$

where

$$g_{20}=-g_{02}=\left(-A_2+i\cdot A_1\right)\xi;\qquad g_{11}=0;\qquad g_{21}=g_{101}\cdot w_{20}.$$

In accordance with the ratio

$$g_{101}=\frac{1}{2}\left(\frac{\partial^2F_1}{\partial x_1\partial x_3}-\frac{\partial^2F_2}{\partial x_2\partial x_3}+i\left(\frac{\partial^2F_1}{\partial x_2\partial x_3}+\frac{\partial^2F_2}{\partial x_1\partial x_3}\right)\right)=\frac{4\xi^2}{1+4\xi^2},$$

and we receive

$$g_{21}=-\frac{4\xi^2\left(\xi+i\cdot\xi^2\right)}{2a\left(1+4\xi^2\right)^2\left(1+\xi^2\right)}.$$

As noted above, when differentiating the eigenvalues with respect to the parameter μ for $\mu=0$, the relation holds: $\operatorname{Re}\lambda'(0)=\alpha'(0)\neq0$. It follows that the conditions of Hopf's theorem are satisfied, and a limit cycle is generated from a complex focus, i.e., an Andronov-Hopf bifurcation occurs.

Let us find the first Lyapunov value, which determines whether this bifurcation is "soft" (safe) or "hard" (dangerous), and, accordingly, whether this limit cycle is stable or unstable [40 et al.]. In this case, the first Lyapunov value has the form:

$$l_1(0) = \frac{g_{21}}{2} + \frac{i}{6\omega}\left|g_{02}{}^2\right|. \tag{21}$$

From expression (21) it follows that

$$\operatorname{Re} l_1(0) = -\frac{\xi^3}{a\left(1+4\xi^2\right)^2(1+\xi^2)} < 0. \tag{22}$$

Fulfilment of condition (22) means that the limit cycle emerging from a complex focus is stable, i.e., the system moves from a stable equilibrium to a state of stable oscillations. A stable focus occurs when $\mu < 0$. If the condition $\mu < 0$ is satisfied, then at the origin there is also a stable focus, but it is not rough. If $\mu > 0$, then the phase trajectories are wound onto a stable limit cycle. Thus, the loss of stability when changing the sign of a small parameter μ occurs with the birth of a stable limit cycle, the radius of which grows as $\mu^{0,5}$. The equilibrium state loses stability and a stable periodic self-oscillation regime arises in the direction $\mu > 0$, the amplitude of which is proportional to the square root of the deviation of the parameter from its critical value. The excitation of self-oscillations is soft.

Based on the general principles of bifurcation theory [41], for this limit cycle we obtain relationships of its main characteristics. Namely:

– the amplitude of the cycle is

$$\varepsilon = \left(-\frac{\operatorname{Re}\lambda'(0)}{\operatorname{Re} l_1(0)}\cdot\mu\right)^{0.5} + 0\left(\mu^2\right); \tag{23}$$

– the period of the cycle is

$$T = \frac{2\pi}{a}\left(1+\tau_2\cdot\varepsilon^2\right) + 0(\varepsilon^4), \tag{24}$$

where

$$\tau_2 = \frac{1}{a}\left(\frac{\operatorname{Im}\lambda'(0)}{\operatorname{Re}\lambda'(0)}\operatorname{Re} l_1(0) - \operatorname{Im} l_1(0)\right).$$

We write the periodic solution itself in the form:

$$x_1 = \operatorname{Re} z; \qquad x_2 = \operatorname{Im} z; \qquad x_3 = \operatorname{Re}\left(w_{20} \cdot z^2\right);$$
$$z = \varepsilon \cdot \exp\left(\frac{2\pi i \cdot t}{T}\right) + \frac{i \cdot g_{02}\varepsilon^3}{6a}\left(\exp\left(-\frac{4\pi i \cdot t}{T}\right) - 3\exp\left(\frac{4\pi i \cdot t}{T}\right)\right) + 0(\varepsilon^3). \tag{25}$$

Let's highlight the advantages of our proposed model. These advantages lie in the fact that the mechanism for the emergence of the innovation cycle under consideration possesses a number of characteristic features that cannot be explained within the framework of the linear theory of ecosystem development. Firstly, oscillatory processes in form (15) are not symmetrical, i.e., the rise and fall may have different durations. Secondly, the cycle amplitude depends on internal factors and is not explained by the influence of external factors.

An example of simulation modeling of cyclic processes that arise in an ecosystem during the diffusion of innovations are the results presented in the work [42]. Based on a nonlinear mathematical model of innovation dynamics, which takes into account the presence of distributed delay (time lag), a study was conducted on the behavior of an ecosystem in a small neighborhood of stationary points in discrete time mode. Depending on the parameter a, which is responsible for the consistency of the innovation diffusion process, and its inverse value characterizes the potential capabilities of the system (availability of resources, technological capabilities), the regimes of stationary development of diffusion processes were identified, but the existence of critical regimes was also discovered, in which self-oscillations arise in the system, and a transition to a state of chaos may occur. The transition to these modes is associated with the fact that the system's potential is insufficient to maintain its stationary development. If this drawback is not very significant, then upon reaching the equilibrium point the system subsequently switches to an oscillatory mode (Fig. 4). Relatively quickly, the amplitude of these oscillations decreases to a constant value, which then persists indefinitely.

The designations shown in Fig. 4 and further have the following meaning: n corresponds to the number of time intervals for which the calculations were carried out, y_{n+1} is the volume of innovative products at the time $n+1$.

As the parameter responsible for consistency changes from a = 3 to a = 3.5, the amplitude of self-oscillations increases almost fivefold. As resource scarcity continues to increase, the system enters a state of chaos (Fig. 5).

The presented calculation results are in good agreement with the general patterns of nonlinear dynamics of innovation processes. In particular, there is an uneven evolution of the economic system, a periodic change in the phases of cycles and the cycles of development of a complex system. It should be remembered that cyclicity is a common form of behavior for a huge number of dynamic systems of various natures, and to predict their development based on information about their previous states, it is necessary to use the powerful mathematical apparatus of the theory of nonlinear dynamics.

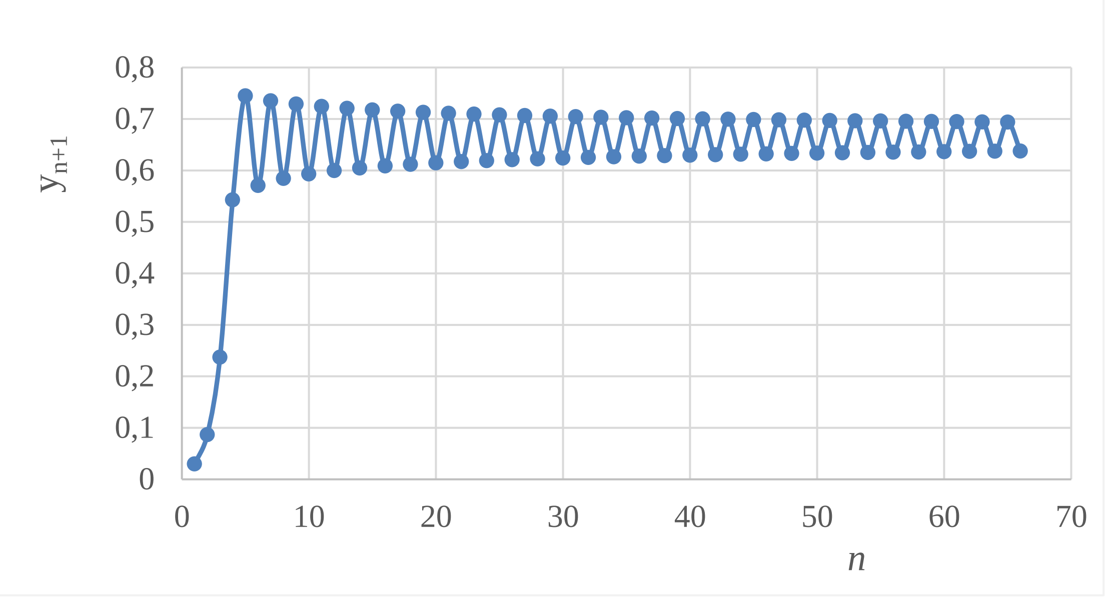

Figure 4. The emergence of self-oscillations during the spread of innovations under conditions of a slight lack of resources when $a = 3$

Source: authors' research results

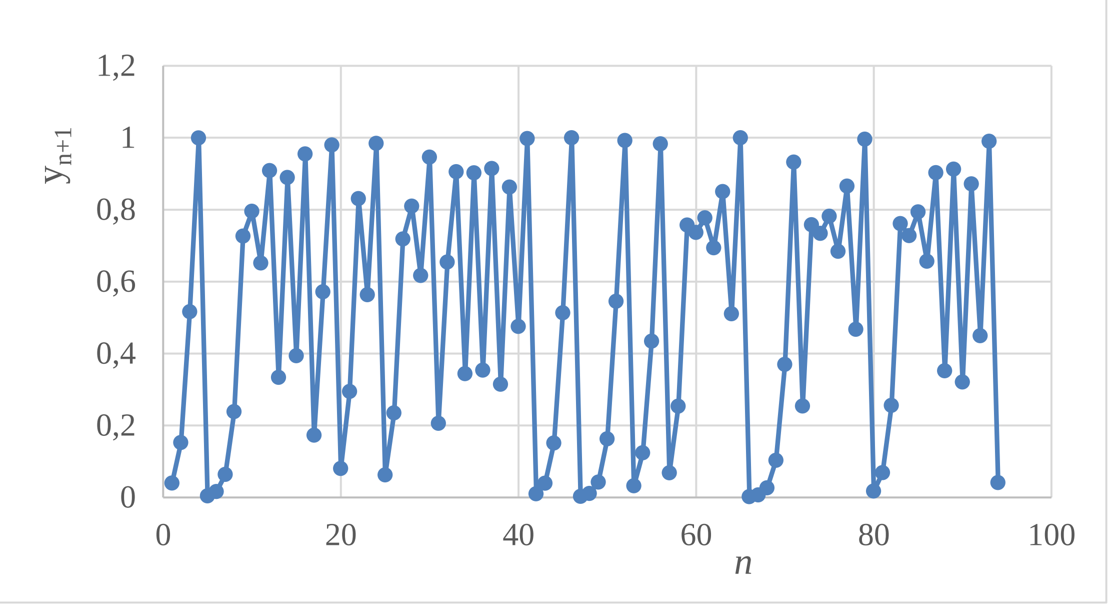

Figure 5. Dynamics of the release of innovative products when $a = 4$

Source: authors' research results

Conclusions

In accordance with modern concepts, the state of the innovation system in the general structure of economic activity is a complex of various subsystems that perform certain technological functions and are interconnected by processes of intensive exchange of material and information flows. By their nature, these innovative systems can be nonlinear, multidimensional and multi-connected, in which complex transient processes can develop with the emergence of critical and chaotic regimes. In light of the implementation of the concept of sustainable development, the tasks of managing this kind of dynamic systems are extremely relevant, but from the point of view of the linear theory of automatic control they are difficult and practically unrealizable.

In this paper, the authors attempted to implement a synergetic approach to the process of managing innovation flows, while the basic idea underlying the construction of a mathematical model of innovation diffusion was to develop a nonlinear model using a built-in regulator. This regulator is a second-order inertial-dynamic link and is the result of a symbiosis of the principles of self-organization and cybernetic control methodology, taking into account the structural features of nonlinear objects and features of their functioning in the presence of both positive and negative feedback. The goal of managing the functioning of such systems with nonlinear feedback was considered to be the achievement of target attractors, that is, asymptotic processes in the space of phase states of the innovation system that correspond to the desired technological regimes of economic objects and ecosystems.

The presence of nonlinear feedback in the system of innovative evolution required conducting research on equilibrium positions for stability. The paper studies in detail the behavior in a small neighborhood of the desired equilibrium volume of an innovative product. It should be noted that the presence of a built-in regulator in a nonlinear structure does not guarantee the stability of the achieved equilibrium level. Based on the proposed mathematical model, the authors of this study formulated the conditions for the stability of a dynamic system and determined the critical values of its parameters. In particular, the existence of stable self-oscillations near the equilibrium position was discovered, and this regime is soft in nature.

The results obtained in this work are consistent with the basic principles of nonlinear oscillation theory, clearly confirming the nonlinearity of the development of innovative systems. The approach to studying the diffusion of innovations based on mathematical models of nonlinear dynamics implemented in this work allows for a more accurate diagnosis of critical states of the ecosystems under study, as well as the search for the most effective ways to overcome them for the successful implementation of an innovative development program.

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